# A Unified Framework for CBDC Design:Remuneration, Collateral Haircuts and Quantity Constraints*

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**Abstract**

We study the macroeconomic effects of central bank digital currency (CBDC) in a dynamic general equilibrium model. Timing and information frictions create a need for money – inside money (bank deposits) and outside money (CBDC) – to be used in financing production. To steer the quantity of CBDC, the central bank can set the lending and the deposit rate for CBDC as well as a collateral or a quantity constraint. Less restrictive provision of CBDC reduces bank deposits whereas a higher lending rate on CBDC can contain bank disintermediation if bank deposits and CBDC are sufficiently close substitutes. A positive interest spread on CBDC as well as stricter collateral or quantity constraints reduce welfare, especially if the elasticity of substitution between bank deposits and CBDC is small.

Keywords: Central bank digital currency, monetary policy, search and matching.

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# 1 Introduction

The digital payments landscape has evolved rapidly over the past years. The emergence of crypto-assets and big tech companies' reflections about issuing private currencies have prompted concerns about safety and data protection related to private currencies. A widespread usage of private currencies could also entail risks to monetary sovereignty and raise financial stability concerns. As a response to these developments, central banks have started their own work programmes to assess the prospects of issuing central bank digital currency (CBDC) for retail transactions. This is documented in a recent survey by the Bank for International Settlements (BIS) (Boar and Wehrli, 2021), which found that central banks representing a fifth of the world's population are likely to issue a general purpose CBDC in the next three years – demonstrating a further significant advancement in these projects compared to two earlier surveys (Barontini and Holden, 2019; Boar et al., 2020).

Central banks' motivations for issuing CBDC derive from its potential to provide a secure, efficient and universally accessible means of payment for everybody (European Central Bank, 2020; Bank for International Settlements, 2020). Beyond this primary motivation, however, the existence of CBDC entails important implications for monetary policy implementation, monetary policy transmission and financial stability.[1] These implications depend on the specific design features of CBDC, such as remuneration, holding limits or the choice and the pricing of assets held against CBDC. These parameters may be set by the central bank in such a way that undesired consequences for monetary transmission and financial stability are mitigated, see e.g. Bindseil (2020); Bindseil and Panetta (2020). Little is known, however, about the effectiveness of these parameters in steering the demand for CBDC and the resulting macroeconomic effects; in particular the implications that variations of them may have on equilibrium allocations and welfare.

To study these questions, we construct a general equilibrium model in the spirit of Lagos and Wright (2005) with search and matching frictions, which require entrepreneurs to borrow inside and outside money. These two types of money – inside money in the form of bank deposits and outside money in the form of CBDC – are needed to pay for two different kinds

1 The literature on monetary policy and financial stability effects of CBDC is evolving quickly and includes Pollock (2018); Carstens (2019); Mancini-Griffoli et al. (2018); Agur et al. (forthcoming).

of inputs that are needed for production. The setup can be motivated by preferences for specific forms of payment on part of the workers who produce these inputs. Some workers, for instance, may hold bank accounts that allow them to access a range of bank services and they therefore prefer deposits although these may offer less protection of private payment data. Other workers, in contrast, may prefer the security of CBDC, which is a liability of the central bank although it may not give access to services that banks typically provide for their customers. Alternatively, this modelling setup can be motivated by spatial separation, where some workers produce and sell their inputs (goods or services) in locations where access to bank deposits is limited whereas others operate in an environment in which they dislike accepting CBDC.[2] In our model, we do not consider cash. The model reflects a setup where CBDC exists in equilibrium, as postulated by the production function, while the macroeconomic effects of different central bank policies with respect to the parameters governing CBDC supply can be studied.

The model comprises three markets that operate sequentially: a settlement, a loan and an investment market. In the settlement market, which constitutes a centralised market, banks, firms, and workers produce and consume a generic good and settle borrowing contracts. Entrepreneurs live for one period only and are born during the loan market with an investment opportunity and a small endowment. Each period, entrepreneurs receive an idiosyncratic shock that determines the optimal size of their production. To realise the projects they need capital goods that are produced by workers. As entrepreneurs cannot commit to repay debt, workers in the investment market require payment in the form of a generally accepted medium of exchange; a friction that gives rise to the demand for money. Therefore, in the loan market the entrepreneurs borrow inside and outside money from commercial banks and from the central bank. But, because of the limited commitment friction, they are required to pledge collateral.[3]

The capital goods produced in the (decentralised) investment market are nonstorable and,

2 As in Duffie et al. (2005) the demand for different monetary assets could also be motivated by different monetary assets generating different flows of disutility or by originating from spatial separation as in Diamond (1982) where production and trade occurs on distinct island economies.

3 In the baseline version of the model we assume that the central bank and commercial banks impose the same collateral requirements. Section 5.2 investigates the case of only the central bank requiring collateral.

therefore, cannot be carried from one market to the next. There are two types of workers who produce capital: A worker of type C only accepts CBDC as payment, whereas type D only accepts commercial bank deposits. An entrepreneur needs to borrow from both a commercial bank and the central bank to be able to buy the inputs needed for production. The central bank chooses the interest rate it charges on loans to entrepreneurs and the interest rate it pays on workers' deposits of CBDC. It can also set a limit on the size of each loan and apply a haircut to future revenue posted as collateral for a loan.

Our modelling approach allows for analysing the impact of the different CBDC design parameters on credit allocation and welfare within a unified framework. Interest rates and haircuts already exist in today's central banks' operational frameworks and are used to influence the demand for central bank reserves (see Sylvestre and Coutinho (2020) for the Eurosystem). With the existence of CBDC the main difference to today's framework would be that also non-banks had to pledge collateral and could earn interest (although potentially at a negative rate) on their holdings of central bank liabilities. A quantity constraint on CBDC loans does not exist today but is one of the safeguards that have been proposed to limit potential effects on monetary policy transmission and financial stability (European Central Bank, 2020; Bank for International Settlements, 2020).[4]

The equilibrium allocation under the first-best solution is characterised by CBDC being unconstrained by collateral requirements or a quantitative cap and by the outside money lending spread being zero.[5] With these CBDC policy parameter settings the central bank can eliminate welfare losses arising from the matching friction in the investment market. If the central bank imposes an interest rate spread between the CBDC lending and deposit rates or collateral or quantity constraints, some investment projects that otherwise would be profitable at market interest rates are not financed, giving rise to inefficiently low investment and consequently production. This result follows from production requiring an input that is best produced with CBDC and costly to substitute. Consequently, welfare gains from CBDC

4 Our model considers a quantity constraint as a measure to prevent a large demand for CBDC at the expense of bank deposits. Quantity constraints are also proposed to address concerns related to money laundering or terrorist financing. These issues, however, are more closely related to the question of privacy for CBDC transactions, which our model does not capture.

5 The spread between the lending and deposit rate for inside money turns out to be irrelevant for the capital allocation and welfare as interest paid on inside money is internalised in a bargaining setup between the firm and the bank.

depend on the degree of substitutability of production inputs paid for in inside or outside money. For increasing degrees of substitution between both forms of money, welfare gains from CBDC decline, approaching zero if both types of money are perfect substitutes.

Besides the effects on the capital allocation and welfare, we are interested in how CBDC design choices affect bank deposits, which we interpret as evidence for the potential of CBDC to disintermediate bank lending. With this notion we want to capture the so-called ”structural” disintermediation as a result of the existence of CBDC (see Bindseil (2020)). If the degree of substitution between commercial bank credit and CBDC is relatively high, a higher interest rate spread and stricter collateral requirements for CBDC increase bank lending. If substitution is low, bank lending falls together with — although less than – CBDC demand as the negative effect on output via the production function dominates. Setting a fixed cap for CBDC loans unambiguously increases the demand for bank loans. Overall, the central bank therefore can contain bank disintermediation by adjusting its policy parameters which, however, can lead to losses in output and welfare.

Within the rapidly expanding literature on CBDC, our approach falls into a class of new-monetarist models assigning an essential role to money on account of search and matching frictions, as in Lagos and Wright (2005).

Accordingly, our approach shares some important commonalities with the model by Keister and Sanches (2019) featuring perfectly competitive banks and credit frictions: CBDC is possibly interest bearing, might crowd out bank deposits, and leads to an increase in welfare by reducing credit frictions. By comparison, in addition to alleviating matching frictions, Chiu et al. (2019) or Andolfatto (2018) derive further welfare improvements from CBDC through reductions in the market power of an imperfectly competitive banking sector. A common feature with these three papers is that in our model investment projects succeed with certainty and all assets are safe. This assumption contrasts with Williamson (2020), where bank deposits mitigate the risk of theft, and Hu and Rocheteau (2015) or Lagos and Zhang (2020), where deposits obscure the risk of the entire asset endowment disappearing with a certain probability.

Yet, there are some important differences: Keister and Sanches (2019) are agnostic about lending or asset purchase operations bringing CBDC into circulation and, accordingly, there is no record keeping associated to CBDC. Furthermore, while in their model firms act both as entrepreneurs and deposit takers, without banks intermediating funds between borrowers and lenders, we distinguish between banks and entrepreneurs, who negotiate on lending volumes and rates, thereby giving rise to a collateral constraint.

Within the class of models with search and matching frictions, Davoodalhosseini (2019) also analyses welfare implications from CBDC, but the model does not feature credit, and welfare changes are rather derived from the central bank being able to accommodate different household preferences with respect to holding cash or CBDC.

An alternative modelling approach to analysing welfare implications from CBDC rests on including money in the utility function originating from Sidrauski (1967): Niepelt (2020) finds that CBDC is neutral with respect to the allocation of capital in the economy. As in Brunnermeier and Niepelt (2019), a shift of deposits into CBDC in the event of a crisis triggers the central bank's lender of last resort role, resulting in a swap of banks' funding sources from private sector deposits to central bank credit. With CBDC resulting in a mere change in the composition of bank funding, resource allocation may not be affected and bank funding may even be more stable. A similar neutrality effect due to an explicit lender of last resort role of the central bank is modelled in Gross and Schiller (2020) who incorporate CBDC in a New Keynesian DSGE model featuring money in the utility function as well.

Our approach also differs from studies based on bank-run models in the tradition of Diamond and Dybvig (1983), showing under which conditions CBDC can either pose a threat to financial stability or reinforce it through more stable bank funding conditions: in Fernandez-Villaverde et al. (2020) the central bank is competing with commercial banks for deposits. As depositors internalise the possibility of a banking panic, CBDC will crowd out bank deposits entirely. The same result is found in Böser and Gersbach (2020) although CBDC, due to the threat of bank runs, enhances monitoring incentives and enforces higher market discipline in the short run.

Without money playing an essential role, DSGE models do not lend themselves to analysing welfare effects from CBDC easily. A prominent example is Barrdear and Kumhof (2016) who model CBDC as an additional short-term central bank liability and issued against longer-term government bonds. Like quantitative easing, such government bond purchases lower the


**Figure 1**
**Market sequencing**

free-float of bonds, thereby sovereign yields, and raise GDP. In this vein, economic benefits arise from a large central bank asset portfolio rather than the change in the composition of the central bank's liabilities.[6] Likewise, in Bitter (2020) macro-financial effects from assets backing CBDC are found to alleviate bank run risk and thereby support financial stability. Ferrari et al. (2020) show that in an open economy DSGE model the introduction of CBDC amplifies international spillovers of shocks.

# 2 The model environment

## 2.1 Market sequencing

There are three types of agents in the model: workers, entrepreneurs, and bankers. Workers and bankers are infinitely lived and have measure 1. Entrepreneurs live for one period only and have measure 1 as well. Time is discrete and the discount factor across periods is $0 < \beta < 1$.

In each period three markets open sequentially (see Figure 1): a *settlement market*, where credit contracts are settled and a generic good is produced and consumed; a *loan market*, where borrowing takes place; and an *investment market*, where production of capital goods takes place. All goods are perfectly divisible and nonstorable, which means that they cannot be carried from one market to the next. There are two perfectly divisible financial assets: deposits (inside money) and central bank digital currency (from now on CBDC; virtual outside money).

In the *settlement market*, a generic good can be produced and consumed by all agents.

6 Accordingly, Kumhof and Noone (2018) suggest to stave off fears of bank disintermediation risk by making CBDC distinct from reserves, bank deposits, and cash. Specifically, they stipulate that CBDC should be issued against eligible securities and be nonconvertible into cash or reserves, thus removing the key feature of what constitutes money. It thereby is conceived to become a separate policy instrument.

They all have the same constant returns to scale production technology with one unit of the good being produced with one unit of labour generating one unit of disutility. Thus, producing $h$ units of goods implies disutility $-h$. Furthermore, for all agents the utility of consuming $x$ units of the generic goods yields utility $x$. As in Lagos and Wright (2005) these assumptions yield a degenerate distribution of portfolios at the beginning of the loan market.[7]

In the *loan market*, entrepreneurs are born with an investment opportunity. To implement their projects they need capital goods that are produced by the workers in the investment market. Since entrepreneurs cannot commit to repay debt, the workers require payment in form of a generally accepted medium of exchange. Thus, this limited commitment friction generates a demand for money. The entrepreneurs borrow money from the central bank and the commercial banks. Due to the limited commitment friction, lenders require collateral. There are two types of collateral: future output $y$ and an endowment $\tilde{e}$ that entrepreneurs receive when they are born in the loan market. Since the entrepreneur can divert a fraction $1 - \bar{\eta}$ of the output $y$, only a fraction $\bar{\eta}$ of future output is available as collateral. The endowment $\tilde{e}$ is idiosyncratic and yields discounted utility $e = \beta\tilde{e}$ to the entrepreneur when consumed in the settlement market. It has no consumption utility to any other agent in the model.

In the loan market entrepreneurs are matched with bankers pairwise at random. Since both types of agents have measure 1 we assume that every banker and every entrepreneur has a match. A banker entrepreneur pair bargains about the size of the loan and the interest rates. The terms of the loan are determined according to a surplus splitting rule.

In the *investment market* capital goods $k$ are produced by workers and bought by the entrepreneurs, with an investment of $k$ yielding

$$y = \beta^{-1}\varepsilon^{1-\alpha}f(k) \tag{1}$$

units of generic goods in the following settlement market. Capital $k = (k_C, k_D)$ consists

7 All agents have the same production technology and the same utility function in the settlement market. However, in equilibrium workers and bankers will consume only and entrepreneurs will sell a fraction of their output for money in this market.

of two different types, $k_C$ and $k_D$ that have to be paid for by the entrepreneur either with CBDC, denoted by the subscript $C$, or with deposits, $D$. The function $f(k)$ satisfies

$$f(k) = \left(ak_C^{\rho} + (1-a)\, k_D^{\rho}\right)^{\alpha/\rho} \tag{2}$$

with $a$ being the input share of $k_D$ and $k_C$ in the production function, $0 < \alpha < 1$ the degree of homogeneity and $\rho \leqslant 1$ the substitution parameter.[8] The term $\varepsilon$ is an idiosyncratic investment shock that becomes known at the beginning of the loan market when entrepreneurs are born. The investment shock has a continuous log-normal distribution $G(\varepsilon)$ with support $(0, \infty)$, the shock is i.i.d. across banks and serially uncorrelated. The investment shock generates a distribution of investment projects that allows us to study how CBDC design affects firms or investment projects of different size in the economy.

The capital goods are produced and sold by workers at competitive prices on two distinct islands. A fraction of the workers can produce $k_C$ and lives on island $C$, the remaining fraction can produce $k_D$ and lives on island $D$. The precise measure of workers on these islands is irrelevant since we will specify preferences and technologies such that the workers are indifferent with regard to how much to produce.

Holding money $i$ provides the flow disutility $\varphi_{ij}$ to workers living on island $j$.[9] We assume that workers that can produce $k_C$ dislike inside money but are neutral towards CBDC:

$$\varphi_{CC} = 0 \text{ and } \varphi_{DC} \ll 0.$$

In contrast, workers that can produce $k_D$ dislike CBDC and are neutral towards inside money:

$$\varphi_{CD} \ll 0 \text{ and } \varphi_{DD} = 0.$$

These flow utilities generate a demand for both assets even when the rates of return of the two monies are not equal.[10] Nevertheless, in our model the flow utilities are irrelevant for

8 We assume that the project succeeds with certainty. However, this would be easy to relax.
9 Similar assumptions are made in the finance search and matching literature pioneered by Duffie et al. (2005).
10 This also allows us to circumvent the famous Kareken and Wallace (1981) indeterminacy result.

investment decisions when $k_C$ and $k_D$ are perfect substitutes, i.e. $\rho \to 1$. In this case, the two monies can coexist only if they have the same rate of return because entrepreneurs will exclusively use the type of money that is less costly for them.

In this paper, we assume that $\varphi_{DC}$ and $\varphi_{CD}$ are sufficiently large negative numbers such that on island $j$ workers only accept money $j = C, D$.[11] Note further that we assume that each worker on an island produces the same amount and each bank has the same customer base on island $D$. This implies that banks have a symmetric inflow of inside money at the end of the investment market.

## 2.2 Money supply

The central bank maintains two standing facilities. At the borrowing facility entrepreneurs and bankers can borrow CBDC against collateral. The central bank sets the collateral haircut $\eta$ so that $\eta \leqslant \bar{\eta}$ and the lending rate $i_{\ell_C}$. The central bank can also impose a quantity constraint on CBDC loans in form of a maximum loan amount, $\bar{k}$. In the baseline version of the model we assume that the commercial banks set the same haircut as the central bank, whereas in section 5.2 we investigate the case when the commercial banks do not request collateral. At the deposit facility all agents can deposit CBDC and earn the deposit rate $i_C$. The deposit facility is a form of 'central bank accounts for all' and we assume that all entrepreneurs and workers have such an account.[12] In equilibrium, the workers hold the CBDC overnight and, therefore, earn $i_C$ on their CBDC holdings.

The stock of CBDC is completely endogenous in the model. When an entrepreneur receives a loan from the central bank, CBDC is created. When the loan is redeemed, CBDC is destroyed.[13] Let $M_C$ denote the aggregate quantity of CBDC at the beginning of the settlement market. Note that

$$M_C = \int_0^\infty \ell_C(\varepsilon) dG(\varepsilon),$$

11 An interesting extension of the model would assume that $\varphi_{DC}$ and $\varphi_{CD}$ are small, but strictly negative numbers. In this case, the rate of return of the two monies can be different without driving out one of the monies as long as the difference is not too large.

12 Entrepreneurs and workers on island $D$ also have inside money accounts at the commercial banks. They have access to the respective accounts if they are on the respective island and in the settlement market.

13 For the same reasons, the aggregate stock of inside money is endogenous as well.

where $\ell_C(\varepsilon)$ is the CBDC loan taken out by an entrepreneur that drew the idiosyncratic investment shock $\varepsilon$. The real budget constraint of the central bank satisfies

$$T = M_C \phi_C (i_{\ell C} - i_C).$$

We assume that the central bank hands out any profit or loss in form of lump sum transfers to the agents in the model. Without loss of generality, we assume that the transfers go to the entrepreneurs. Thus, since the measure of entrepreneurs is 1, each entrepreneur receives $T$.

### 2.3 First-best allocation

In the Appendix we derive the first-best investment quantities that would be attained in the absence of the limited commitment friction. Optimality requires that the marginal return from an investment project equals the marginal cost of capital. That is,

$$\varepsilon^{1-\alpha} f_j(k_j) = 1,\ j = C, D. \tag{3}$$

The optimal ratio satisfies

$$\omega^* \equiv \frac{k_C^*}{k_D^*} = \left(\frac{a}{1-a}\right)^{\frac{1}{1-\rho}}. \tag{4}$$

Closed-form solutions for $k_C^*$, $k_D^*$, and output $y^*$ are stated in the Appendix. Note that $\beta y^* - k_C^* - k_D^* \geqslant 0$ for all $\varepsilon$. The implication is that from a societal point of view all projects should be implemented. We will compare the market allocation with this first-best solution. Furthermore, we will discuss how CBDC design can affect the market solution and bring it closer to the first best.

## 3 Decision problems

In this section, we study the decision problems of all agents in the three markets. We start with the investment market, where workers decide how much capital to produce and sell. Then, we move to the settlement market, where output is consumed, loans are settled, and banks decide how much money they bring to the loan market. Finally, we describe the loan market, where entrepreneurs take decisions on investment and loans, negotiating the latter with banks.

## 3.1 Investment market

Let $p = (p_C, p_D)$ denote the prices of the two capital goods. On island $j = \{C, D\}$, workers produce capital goods $k_j$ with linear disutility $c(k_j) = k_j$ and then sell the goods at the competitive price $p_j$ to the entrepreneurs for money $j$. They hold the money overnight in their accounts, earning interest on it. In the settlement market, they spend the money on the generic good obtaining linear utility $U(x) = x$. It is straightforward to show that workers are indifferent as to how much they sell in the investment market if

$$p_j \beta \phi_j^+ (1 + i_j) = 1, \tag{5}$$

where $\phi_j^+$ is the value of money $j$ in next period's settlement market and $i_j$ is the interest rate paid for owning money $j$ overnight. Since workers are indifferent with regard to how much they produce, we assume that each worker on an island produces the same amount, which implies that each bank has the same customer base. Therefore, banks have a symmetric inflow of inside money at the end of the investment market.

## 3.2 Settlement market

### 3.2.1 Banks

Let $B_S(m, \ell)$ denote a bank's expected value of entering the settlement market with money holdings $m = (m_C, m_D)$ and loans $\ell = (\ell_C, \ell_D)$. The quantity $m_D$ is a liability to the bank (deposits) and $m_C$ is a liability of the central bank (central bank deposits). In contrast, $\ell_D$ is an asset for the bank (the loan extended to the entrepreneur) and $\ell_C$ is a liability of the commercial bank (borrowing from the central bank).

The marginal values of holding the assets at the beginning of the settlement market are

$$B_S^D = -\phi_D (1 + i_D) \text{ and } B_S^C = \phi_C (1 + i_C), \tag{6}$$

$$B_S^{\ell_C} = -\phi_C (1 + i_{\ell_C}) \text{ and } B_S^{\ell_D} = \phi_D (1 + i_{\ell_D}), \tag{7}$$

where $B_S^j$ is the partial derivative of $B_S$ with respect to $m_D$ and $m_C$, respectively, and $B_S^{\ell_j}$ is the partial derivative of $B_S$ with respect to $\ell_D$ and $\ell_C$, respectively. Consider, for example, the derivative $B_S^D$. The banker pays the interest rate $i_D$ and thus the marginal debt is $(1 + i_D)$. The value of deposits in terms of the generic good is $\phi_D$. That is, $(1 + i_D)$ deposits buy $\phi_D (1 + i_D)$ generic goods. Given the banker's linear consumption preferences, the disutility of redeeming a marginal unit of deposits is $-\phi_D (1 + i_D)$. The interpretation of the other derivatives is similar.

Note that bankers bring no money into the loan market. The reason is that they can produce inside money on the spot by extending loans to entrepreneurs. Furthermore, they hold no CBDC because the entrepreneurs can borrow CBDC directly at the central bank.

### 3.2.2 Workers

Workers produce capital goods in the investment market and sell them for deposits or CBDC, for which they earn interest payments in the beginning of the settlement market. Because of the linearity of their preferences for consumption and leisure (disutility of working), they spend all their money holdings for units of the generic good and move into the following loan and investment markets without any money holdings. For workers the marginal values of bringing an additional unit of CBDC or inside money into the settlement market are

$$W_S^D = \phi_D (1 + i_D) \text{ and } W_S^C = \phi_C (1 + i_C). \tag{8}$$

### 3.2.3 Entrepreneurs

$E_S (y, m, \ell)$ denotes an entrepreneur's expected value of entering the settlement market with goods $y = \beta^{-1}\varepsilon^{1-\alpha} f(k)$, money holdings $m = (m_C, m_D)$, and loans $\ell = (\ell_C, \ell_D)$. Here, $\ell_D$

and $\ell_C$ are liabilities and $m_C$ and $m_D$ are assets. The marginal values of holding these assets at the beginning of the settlement market are

$$E_S^y = 1, \tag{9}$$

$$E_S^C = \phi_C (1 + i_C) \text{ and } E_S^D = \phi_D (1 + i_D), \tag{10}$$

$$E_S^{\ell_C} = -\phi_C (1 + i_{\ell_C}) \text{ and } E_S^{\ell_D} = -\phi_D (1 + i_{\ell_D}). \tag{11}$$

The marginal value of the assets have the same interpretation as above. The marginal value of bringing generic goods into the settlement market is 1 because the marginal disutility of producing generic goods in the settlement market is 1.

An $\varepsilon$-entrepreneur who has invested $k = (k_C, k_D)$ holds $y$ units of the production good at the beginning of the settlement market. To repay the loans, the entrepreneur sells some of them for CBDC and some for inside money to repay the loans and consumes the rest. Upon repaying the loans the entrepreneur receives back the collateral $\tilde{e}$, consumes it yielding utility $e$, and leaves the economy.

## 3.3 Loan market

In the loan market entrepreneurs borrow CBDC and inside money in order to pay for the capital goods in the investment market. Entrepreneurs cannot commit to repay loans. For this reason, the central bank and the commercial bank require collateral.

For an $\varepsilon$-entrepreneur who borrows $\ell$ the indirect utility function $E_L(\varepsilon)$ satisfies

$$E_L(\varepsilon) = \beta E_S (y, m, \ell),$$

where $m = \ell - pk$ is the amount of money left after spending $pk$ units on capital. Since the entrepreneur has linear utility in the settlement market and, in equilibrium, pays back the loan, receiving back his collateral, we have

$$\beta E_S (y, m, \ell) = e + \varepsilon^{1-\alpha} f(k) - \beta \phi_C \ell_C (1 + i_{\ell_C}) - \beta \phi_D \ell_D (1 + i_{\ell_D}).$$

This expression shows that if borrowing rates exceed deposit rates, the entrepreneur will always spend all money on capital goods; that is, if $i_{\ell_C} \geqslant i_C$ and $i_{\ell_D} \geqslant i_D$, then $m_C = \ell_C - p_C k_C = 0$ and $m_D = \ell_D - p_D k_D = 0$.[14] For the rest of this paper we will assume that $i_{\ell_j} \geqslant i_j$, implying $\ell_j = p_j k_j$ for $j = \{C, D\}$.

For a banker matched with an $\varepsilon$-entrepreneur, the indirect utility function $B_L(\varepsilon)$ satisfies

$$B_L(\varepsilon) = \beta B_S\left(m_D, \ell_D\right).$$

This expression takes into account that the bankers enter the loan market with no money and no liabilities. If they extend a loan $\ell_D$ to the entrepreneurs, they create inside money $m_D = \ell_D$. Note again that $m_D$ is a liability. Using the linearity of preferences in the settlement market, we can write this expression as follows:

$$\beta B_S\left(m_D, \ell_D\right) = \beta \phi_D^+ \ell_D (1 + i_{\ell_D}) - \beta \phi_D^+ m_D (1 + i_D) + E B_L(\varepsilon).$$

This expression includes a continuation value $EB_L(\varepsilon)$ because the bankers are infinitely lived and will be matched in the future with entrepreneurs drawn from the distribution $G(\varepsilon)$, which is reflected in the expectation operator.

### 3.3.1 Bargaining

In the loan market, a banker-entrepreneur pair negotiates the loan quantities $\ell_D$ and $\ell_C$ and the interest rate on the inside money loan $i_{\ell_D}$. After the negotiation the entrepreneur receives $\ell_D$ from the bank and borrows the required amount $\ell_C$ directly from the central bank at the given policy rate $i_{\ell_C}$.

We assume that bargaining follows a bargaining splitting rule. That is, bankers maximise their payoff subject to the constraint that the entrepreneurs receive at least a fraction $1 - \theta$ of the total surplus $TS = S_B + S_E$, which is the sum of the individual surpluses. In what follows we derive these surpluses and the bargaining solution.

14 If $i_{\ell_C} = i_C$ and $i_{\ell_D} = i_D$, entrepreneurs are indifferent on how much to borrow. Without loss of generality, we assume that in this case they only borrow what they will spend on capital goods.

The banker's surplus from the loan is

$$S_B = \beta B_S(m_D, \ell_D) - \beta B_S(0,0),$$

where $\beta B_S(0,0)$ is the banker's utility if there is an exogenous breakdown of the negotiation. Exploiting the linearity of payoffs in the settlement market and the workers' first-order condition in the investment market, (5), the banker's surplus satisfies

$$S_B = k_D \frac{(i_{\ell_D} - i_D)}{1 + i_D}. \tag{12}$$

The entrepreneur takes out a CBDC loan at the central bank and an inside money loan from the banker. The entrepreneur's surplus is

$$S_E = \beta E_S\left(y, m, \ell\right) - \beta E_S\left(0,0,0\right),$$

where $E_S\left(0,0,0\right) = e$ is the entrepreneur's utility if there is an exogenous breakdown of the negotiation.[15] The entrepreneur's surplus satisfies

$$S_E = \varepsilon^{1-\alpha} f\left(k_C, k_D\right) - \beta\phi_C \ell_C \left(1 + i_{\ell_C}\right) - \beta\phi_D \ell_D \left(1 + i_{\ell_D}\right).$$

Using the workers' first-order conditions yields

$$S_E = \varepsilon^{1-\alpha} f\left(k_C, k_D\right) - \iota_C k_C - \iota_D k_D,$$

where $\iota_C \equiv \frac{1+i_{\ell_C}}{1+i_C}$ and $\iota_D \equiv \frac{1+i_{\ell_D}}{1+i_D}$. The total surplus satisfies

$$TS = S_E + S_B = \varepsilon^{1-\alpha} f\left(k_C, k_D\right) - \iota_C k_C - k_D. \tag{13}$$

Notice that the total surplus does not directly depend on the loan rate $i_{\ell_D}$. As we show

15 We assume here that in case of an exogenous breakdown of the negotiation, the entrepreneurs produce nothing and just consume their endowment in the settlement market.

further below, the role of the inside money loan rate is to split the total surplus between the banker and the entrepreneur.

The bankers maximise their surplus $S_B$ subject to the condition that the entrepreneurs receive the fraction $1-\theta$ of the total surplus. That is,

$$S_E \geqslant (1-\theta)\, TS. \tag{14}$$

Furthermore, the banker takes into account the limited commitment problem of the entrepreneur via the following collateral constraint:

$$\iota_C k_C + k_D + \theta TS \leqslant e + \eta \varepsilon^{1-\alpha} f\left(k_C, k_D\right). \tag{15}$$

In the Appendix, we derive (15) from basic principles. The left-hand side is the sum of the loans to be repaid, including the interest on the loans since $\iota_D k_D = k_D + \theta TS$. The right-hand side of (15) reflects that the entrepreneur can only pledge the fraction $\eta$ of future output plus the discounted utility of the idiosyncratic collateral $\tilde{e}$. Note that (15) is measured in terms of utility.

The banker also takes into account the quantity constraint on CBDC accounts which is one of the design instruments available to the central bank. The constraint is $\ell_C \leqslant \overline{M_C}$, where $\overline{M_C}$ is a nominal cap, or in real terms,

$$k_C \leqslant \bar{k} \equiv \overline{M_C}/p_C. \tag{16}$$

We assume that the central bank indexes $\overline{M_C}$ to inflation so that $\bar{k}$ is constant.

With these constraints, the banker's optimization problem solves

$$(k_C, k_D) = \arg\max_{k_C, k_D} S_B \text{ s.t. (14) - (16).} \tag{17}$$

# 4 Market equilibrium

We now derive the equilibrium quantities for capital $k_C$ and $k_D$ and, after that, the corresponding interest rates and prices. In this context, a central variable is the size of the investment shock, $\varepsilon$, that determines whether and when the collateral and quantity constraints start to bind. We therefore first investigate the different regions in which these constraints are binding before we turn to the effects of the policy parameters on capital allocation, bank lending rates and welfare.

## 4.1 Collateral constraint and quantitative cap

**Definition 1.** The quantities $k_C$ and $k_D$ that solve (17) are an equilibrium for this economy.

Once we have solved for $k_C$ and $k_D$, we can derive all other real and nominal quantities from the solution of (17). Note first that in any solution, condition (14) holds with equality since it can not be optimal for the banker to provide more surplus to the entrepreneur than necessary. Thus, we can use (14) to replace $S_B$ and obtain

$$(k_C, k_D) = \arg\max_{k_C, k_D} \theta TS \text{ s.t. } (15) \text{ and } (16).$$

Let $\lambda$ be the multiplier for constraint (15) and let $\kappa$ be the multiplier for constraint (16). Then, the first-order conditions for $k_C$ and $k_D$ satisfy

$$\varepsilon^{1-\alpha} f_C = \frac{[\theta + \lambda(1-\theta)]\,\iota_C + \kappa}{\theta + \lambda(\eta - \theta)} \text{ and } \varepsilon^{1-\alpha} f_D = \frac{\theta + \lambda(1-\theta)}{\theta + \lambda(\eta - \theta)}. \tag{18}$$

The two multipliers characterise the equilibrium input quantities. Depending on whether the constraints are binding or not, we can distinguish four regions, as presented in Table 1.

| Regions | Multipliers | Quantities: $k_C$ and $k_D$ solve | | |
|---|---|---|---|---|
| 1 | if $\lambda = 0$ and $\kappa = 0$ | $\varepsilon^{1-\alpha} f_C = \iota_C$ | and | $\varepsilon^{1-\alpha} f_D = 1$ |
| 2 | if $\lambda = 0$ and $\kappa > 0$ | eq. (16) | and | $\varepsilon^{1-\alpha} f_D = 1$ |
| 3 | if $\lambda > 0$ and $\kappa = 0$ | $f_C = f_D \iota_C$ | and | eq. (15) |
| 4 | if $\lambda > 0$ and $\kappa > 0$ | eq. (16) | and | eq. (15) |

**Table 1**
**Multipliers and equilibrium conditions for quantities**

If none of the constraints binds, we are in region 1 with $\lambda = 0$ and $\kappa = 0$ and hence $k_C$ and $k_D$ solve $\varepsilon^{1-\alpha} f_C = \iota_C$ and $\varepsilon^{1-\alpha} f_D = 1$. The other entries in Table 1 can be read in the same way.

Whether the multipliers are binding or not depends on the relative size of the model parameters. Moreover, the size of the investment shock plays an important role, as entrepreneurs that draw a small shock will less likely face constraints whereas for larger shocks the collateral or the quantity constraint may become binding. In the following, we investigate the conditions for the order in which the collateral and the quantity constraints start to bind. Let

$$\psi \equiv (1 - \theta)\,\alpha - (\eta - \theta),$$

and assume for the moment that the central bank does not impose a quantity constraint; i.e. $\bar{k} \to \infty$.

**Lemma 1.** If $\psi < 0$, the collateral constraint never binds while for $\psi > 0$ there exists a critical value $\varepsilon^{13}$ such that the collateral constraint is not binding if $\varepsilon \leqslant \varepsilon^{13}$ and it is binding if $\varepsilon > \varepsilon^{13}$ with $\varepsilon^{13}$ denoting the cut-off value for the regions in Table 1 at which region 1 is exited and region 3 is entered.

The closed-form solution for $\varepsilon^{13}$ can be found in the Appendix.

If $\eta > (1 - \theta)\,\alpha + \theta$ the collateral constraint never binds, meaning that the entrepreneurs are not constrained by their inability to commit because the limited commitment problem is small in the sense that only a small fraction $1 - \eta$ of future output can be diverted. Note that when $\alpha > \eta$, i.e. when the degree of homogeneity exceeds the share of output that cannot be diverted, $\psi > 0$ for any $\theta$. For the rest of the paper we assume that $\psi > 0$ so that the collateral constraint is binding for some investment shocks.

We now turn to the conditions under which the quantity constraint is binding. Let

$$\Phi \equiv \frac{e\alpha}{\psi\left(\iota_C + 1/\omega\right)}, \qquad \text{where } \omega \equiv \frac{k_C}{k_D}.$$

**Proposition 1.** *If*

$$\bar{k} > \Phi, \tag{19}$$

*there exist critical values $\varepsilon^{34} > \varepsilon^{13} > 0$, defined in the proof (with $\varepsilon^{34}$ denoting the cut-off value at which region 3 is exited and region 4 is entered), such that for $\varepsilon \leqslant \varepsilon^{13}$, $\lambda = \kappa = 0$, for $\varepsilon^{13} < \varepsilon \leqslant \varepsilon^{34}$, $\lambda > \kappa = 0$, and, finally, for $\varepsilon > \varepsilon^{34}$, $\lambda, \kappa > 0$.*

Proposition 1 describes a situation where we move from region 1 to region 3 and then to region 4 as we increase $\varepsilon$ from $\varepsilon = 0$. The condition $\bar{k} > \Phi$ ensures that the collateral constraint binds prior to the quantity constraint. For $\bar{k} < \Phi$ we move from region 1 to region 2 and then to region 4. For the rest of the paper we assume $\bar{k} > \Phi$ as this is the more interesting case with regard to the evolution of $k_C$ in regions 1, 3, and 4.[16]

To provide some intuition for these analytical results, we present a numerical example to illustrate how the different constraints start to bind as $\varepsilon$ increases. We use the parameter values shown in Table 3, which imply that with an increasing $\varepsilon$ we move from region 1 to region 3 and finally to region 4.

16 In the alternative case the binding quantity constraint would keep $k_C$ constant over regions 2 and 4, leading to a change in the slope for the path of $k_D$ only.

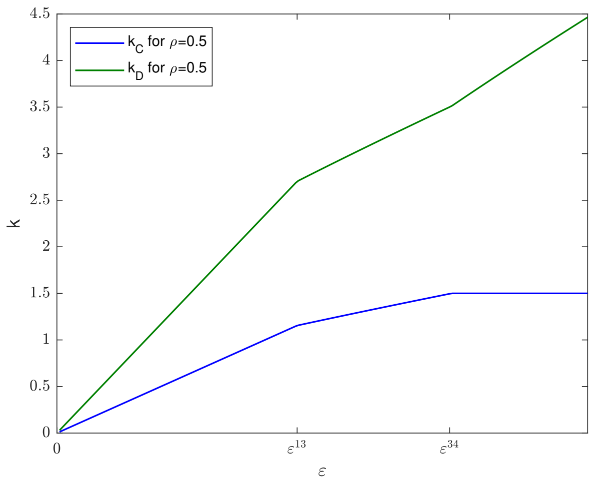



**Figure 2**
**Optimal capital investment**

Figure 2 plots the optimal quantities $k_C$ and $k_D$ as a function of the idiosyncratic investment shock, $\varepsilon$. Because of the value of $a$ that determines the input shares in the production function and the positive spread $\iota_C$ on CBDC, entrepreneurs in this example demand a higher quantity of inside money than of CBDC. With a larger $\varepsilon$ the optimal quantities of $k_C$ and $k_D$ increase. From the threshold $\varepsilon^{13}$ on, the collateral constraint starts to bind but not yet the quantity constraint. At this point, $k_C$ and $k_D$ still increase in $\varepsilon$ but less strongly since only part of the higher output that results from a larger $\varepsilon$ can be pledged as collateral. The amount of invested $k_C$ attains its maximum at $\varepsilon^{34}$ as the quantity constraint starts to bind. Independent of a further increase in $\varepsilon$, $k_C$ is now capped and equal to $\bar{k}$. At the same time, $k_D$ rises with a steeper slope because all additional investment resulting from a higher investment shock is now paid for with inside money.

## 4.2 Nominal equilibrium quantities

So far we have derived the real quantities $k_C$, $k_D$, $y$ and the threshold values for the transition between the different regimes. It remains to determine the prices $p_C, p_D, \phi_C$, and $\phi_D$ and the interest rates $i_D$ and $i_{\ell_D}$. Note that in any equilibrium all prices must grow at the same rate $\gamma$:

$$\gamma \equiv \frac{p_C^+}{p_C} = \frac{\phi_C}{\phi_C^+} = \frac{p_D^+}{p_D} = \frac{\phi_D}{\phi_D^+}. \tag{20}$$

We assume that the central bank controls $\gamma$. In what follows we show how the prices and the interest rates are determined.

First, from the workers' arbitrage condition for inside money, (5), we have

$$p_D \beta \phi_D^+ (1 + i_D) = 1.$$

Using (20), this equation determines the effective price of $k_D$:

$$p_D \phi_D (1 + i_D) = \frac{\gamma}{\beta}. \tag{21}$$

The product $p_D (1 + i_D)$ is the effective price of $k_D$ from a worker's point of view. It is the amount of inside money that is available in the settlement market to a worker who has produced and sold one unit of $k_D$ in the investment market to the entrepreneur against a payment of $p_D$, and the quantity $p_D i_D$ is obtained in the settlement market from the banker who issued the unit of money. The term $p_D \phi_D (1 + i_D)$ is the real quantity of inside money obtained from producing one unit of capital. This arbitrage equation also shows that the effective price of $k_D$ does not change in $i_D$ for a given value of inside money $\phi_D$. An increase in $i_D$ lowers only $p_D$ as inside money becomes more valuable to workers so that they are willing to work more for one unit, keeping the product $p_D (1 + i_D)$ constant.

Second, from the workers' arbitrage condition in the CBDC market, (5), we have

$$p_C \beta \phi_C^+ (1 + i_C) = 1.$$

This equation determines the effective price of $k_C$:

$$p_C \phi_C (1 + i_C) = \frac{\gamma}{\beta}. \tag{22}$$

The product $p_C (1 + i_C)$ is the effective price of $k_C$, following the same intuition like for the effective price of $k_D$. There is one important difference, however, because the interest rate $i_C$ is a policy variable and, hence, exogenous. Note that an increase in $i_C$ will lower the price of $k_C$, $p_C$, for a given value of CBDC, $\phi_C$. The intuition is the same as for an increase in $i_D$.

Third, inside money is a promise to pay central bank money, i.e. CBDC in our model, on demand. We assume that commercial banks keep their promises and, hence,

$$\phi_D = \phi_C.$$

Fourth, this allows us to combine (21) and (22) as follows

$$p_C (1 + i_C) = p_D (1 + i_D). \tag{23}$$

Fifth, we can normalize one price since the initial price level is indeterminate. That is, we set

$$p_C(t = 0) = 1.$$

This choice determines the path for the product $p_D (1 + i_D)$, although the individual components $p_D$ and $i_D$ are not yet determined.

Sixth, we can rewrite (12) as follows:

$$\frac{(i_{\ell_D} - i_D)}{1 + i_D} = \theta TS/k_D. \tag{24}$$

The right-hand side of this equation depends only on $k_C$ and $k_D$ which are determined by equilibrium conditions and therefore pin down $\frac{(i_{\ell_D} - i_D)}{1+i_D} = \iota_D - 1$. Note that $\iota_D$ depends on $\varepsilon$ since $\theta TS/k_D$ may depend on $\varepsilon$.

Finally, the price path for $p_D$ and the two interest rates $i_{\ell_D}$ and $i_D$ still need to be determined, but there are only two equations (23) and (24). Thus, although all real quantities are uniquely determined, the model has a nominal indeterminacy. Rather than adding a theory of how the deposit rate is determined to remove this nominal indeterminacy, we investigate the model under two alternative assumptions:

Assumption 1: We assume that the inside money deposit rate $i_D = 0$, which is in line with the level of deposit rates in developed countries for the last couple of years (see, for example, Berentsen et al. (2020)).

Assumption 2: We assume that the inside money deposit rate equals the CBDC deposit rate, $i_D = i_C$, which could be motivated by the existence of arbitrageurs that can profitably exploit any deviation from $i_D = i_C$.

All real quantities and welfare are independent from whether we use Assumption 1 or 2 and only the results for the bank lending rate, $i_{\ell_D}$, are affected. In the following, we therefore assume that $i_D = 0$ so that the spread $\iota_D$ can be interpreted as the bank lending rate. We discuss in section 6.4 how the bank lending rate $i_{\ell_D}$ changes if we instead assume that $i_D$ follows $i_C$.

# 5 Optimal CBDC design: remuneration and haircuts

In this section we first compare the market solution of the model to the first-best allocation and derive the optimal CBDC design. We then investigate the effects of different rates of remuneration and collateral haircuts analytically. We also consider whether the introduction of CBDC drives out inside money and hence leads to bank disintermediation. In section 6, we present numerical simulations of the model and use them to further discuss the tools for an optimal CBDC design.

**Corollary 1.** For any $\varepsilon \leqslant \varepsilon^{34}$, the ratio $k_C/k_D$ satisfies

$$\omega \equiv \frac{k_C}{k_D} = \left(\frac{a}{\iota_C\,(1-a)}\right)^{\frac{1}{1-\rho}} \tag{25}$$

and the policy rate $\iota_C$ has a direct effect on capital investments.

The first-best allocation as described in (3) requires that the ratio of $k_C$ to $k_D$ is constant and satisfies $\omega^*$. From (25), the central bank can achieve this ratio for any $\varepsilon \leqslant \varepsilon^{34}$ by setting

$$\iota_C = 1.^{17}$$

Note that the first-best allocation is still not attained because for entrepreneurs with an $\varepsilon > \varepsilon^{13}$ the collateral constraint binds in the market solution. The underlying friction

17 This is the minimum value for $\iota_C$. If the central bank were to set the spread to negative values, the entrepreneur would want to borrow money and deposit it with the central bank instead of investing. We rule this out by assumption.

cannot be solved by the central bank although a lowering of collateral haircuts would allow entrepreneurs to borrow more and would move the market allocation closer towards the first-best allocation.[18]

## 5.1 Remuneration, bank intermediation and bank lending rates

We now analyse how a change in the policy rate affects bank intermediation and bank lending rates. In the Appendix, we show that $\frac{dk_C}{d\iota_C} < 0$ when the quantity constraint does not bind, meaning that with an increase in the CBDC lending rate the central bank can depress the demand for CBDC. We know from (25) that an increase in $\iota_C$ decreases the ratio $k_C/k_D$, which can be interpreted as a higher policy rate on CBDC reducing bank disintermediation but it needs to be checked whether $k_D$ increases in absolute terms. In the Appendix, we derive an explicit solution for $k_D$ when the quantity constraint does not bind and show that

$$\frac{dk_D}{d\iota_C} = -\frac{(\alpha - \rho)\, k_D a \omega^{\rho}}{(1-\alpha)\left[a\omega^{\rho} + 1 - a\right](1-\rho)\, \iota_C}.$$

Thus, if $\rho \lesseqgtr \alpha$, $\frac{dk_D}{d\iota_C} \lesseqgtr 0$, i.e. if the two inputs are sufficiently close substitutes in the sense that $\rho > \alpha$, an increase in $\iota_C$ increases $k_D$. In this context, we can think of a high $\rho$ as CBDC features that makes its functionality close to those of bank deposits. In contrast, if $\rho < \alpha$, an increase in $\iota_C$ decreases $k_D$.

Turning to the effect of $\iota_C$ on bank lending rates, $\iota_D$, we use the explicit solution for $\iota_D$ derived from the banker‘s surplus, as outlined in more detail in the Appendix, and obtain

$$\frac{d\iota_D}{d\iota_C} = \theta \frac{1-\alpha}{\alpha} \frac{\rho}{\rho - 1} \omega < 0 \ .$$

In table 2 we summarise the effects of changes in the policy rate on bank intermediation for different degrees of substitutability.

18 Note that we assume in the baseline version of model that the central bank and commercial banks impose the same collateral requirements. Section 5.2 investigates the case where commercial banks do not require collateral.

| | $\frac{dk_D}{d\iota_C}$ | $\frac{dk_C}{d\iota_C}$ | $\frac{d\iota_D}{d\iota_C}$ |
|---|---|---|---|
| $\rho > \alpha$ | $+$ | $-$ | $-$ |
| $\rho < \alpha$ | $-$ | $-$ | $-$ |

**Table 2**
**Effects of the policy rate on bank intermediation**

Section 6 shows simulations of aggregate capital and the bank lending rate as a function of $\iota_C$ for $\rho \lesseqgtr \alpha$.

## 5.2 Collateral haircuts and bank intermediation

The impact of a change in the collateral haircut on bank intermediation can be seen from the first-order conditions (18). If the collateral constraint is binding, an increase in $\eta$, i.e the share of output that cannot be diverted, relaxes the constraint and decreases the marginal products $f_C$ and $f_D$, which results in an increase in $k_D$. The evolution of capital and bank lending rates as a function of the collateral haircut is simulated in section 6.

We can also analyse how results change when haircuts applied by the central bank differ from those of commercial banks. If only the central bank requires collateral, the decision problem changes to[19]

$$\begin{aligned} \left(k_C^N, k_D^N\right) &= \arg\max_{k_C, k_D} \theta\left[\varepsilon^{1-\alpha} f\left(k_C, k_D\right) - \iota_C k_C - k_D\right] \\ \text{s.t.} \quad \iota_C k_C &\leqslant e + \eta \varepsilon^{1-\alpha} f\left(k_C, k_D\right). \end{aligned} \tag{26}$$

The first-order conditions for $k_D$ and $k_C$ would then satisfy

$$\varepsilon^{1-\alpha} f_C = \frac{\iota_C\left(\theta+\lambda\right)}{\theta+\lambda\eta} \text{ and } \varepsilon^{1-\alpha} f_D = \frac{\theta}{\theta+\lambda\eta}. \tag{27}$$

If the collateral constraint does not bind, $k_C$ and $k_D$ solve

$$\varepsilon^{1-\alpha} f_C = \iota_C \text{ and } \varepsilon^{1-\alpha} f_D = 1$$

19 For simplicity, we assume here that the quantity constraint does not bind.

By contrast, if the collateral constraint is binding, $k_C$, $k_D$ and $\lambda$ solve (26) and (27).

Note that the ratio $k_C/k_D$ then satisfies

$$\frac{k_C}{k_D} = \left( \frac{a}{(1-a)} \frac{\theta}{\iota_C(\theta+\lambda)} \right)^{\frac{1}{1-\rho}}.$$

When assuming that only the central bank requires collateral, the ratio thus depends on the policy rate $\iota_C$. Note that we can then define the implicit interest rate charged by the central bank as

$$\iota_C^{\lambda} = \frac{\iota_C(\theta+\lambda)}{\theta},$$

which includes the costs arising from the interest rate $\iota_C$ and the collateral constraint.[20]

# 6 Simulations

To illustrate the mechanics of the model in more detail and to elaborate further on the questions outlined above, we simulate how the key quantities depend on the size of the investment shock, $\varepsilon$, and show the effects of the collateral and the quantity constraints. Since these effects depend on the degree of substitutability, $\rho$, we generally show simulations for two different values of $\rho$. Furthermore, we illustrate how the central bank can steer the demand for CBDC using different policy instruments, thereby also affecting welfare. After presenting simulations of the evolution of capital and bank lending rates as a function of the investment shock, $\varepsilon$, we aggregate these variables across the entire distribution of $\varepsilon$ and show how the use of different policy tools affects the evolution of capital, bank lending rates and welfare. Finally, in section 6.4, we discuss how the results change if we assume that $i_D = i_C$ instead of setting $i_D = 0$ as is done for the simulations in sections 6.1 to 6.3.

For the following simulations we use the parameter values as shown in Table 3 (unless stated otherwise) with a log-normal distribution for the investment shock. The discount factor $\beta = 0.96$ reflects an interpretation of a period as a year, which is the time an investment project needs to mature. Because of the nominal indeterminacy, the model solution does not

20 Alternatively, it could be assumed that the central bank applies larger collateral haircuts than the private sector.

| Parameter | | Value |
|---|---|---|
| Mean and variance of log-normal | $\mu_\varepsilon$ | 1.00 |
| investment shock distribution | $\sigma^2_\varepsilon$ | 2.00 |
| Discount factor | $\beta$ | 0.96 |
| Degree of homogeneity | $\alpha$ | 0.65 |
| Share of banker's surplus | $\theta$ | 0.05 |
| Input share for CBDC | $a$ | 0.40 |
| Endowment | $e$ | 1.00 |
| Substitution parameter | $\rho$ | 0.50 |
| Spread on CBDC | $\iota_C$ | 1.02 |
| Deposit interest rate | $i_D$ | 0.00 |
| Output share that cannot be diverted | $\eta$ | 0.50 |
| Quantity constraint for CBDC | $\bar{k}$ | 1.50 |

**Table 3**
**Parameter values for simulations**

depend on the specific level of deposit or lending rates but only on the spread between them, i.e. $\iota_C$ and $\iota_D$. To anchor the inside money spread, we assume for the following results that deposits are not remunerated, i.e. $i_D = 0$, such that $\iota_D$ equals the loan rate on inside money. We set $\iota_C = 1.02$, which corresponds to an interest rate spread of 2 percentage points at the central bank, equalling the ECB's corridor width before 2009. An increase in $\iota_C$ can thus either be interpreted as a decrease in the CBDC deposit rate, $i_C$ or an increase in the CBDC lending rate, $i_{\ell_C}$. For the collateral constraint, we choose $\eta = 0.5$.[21] We set the input share for CBDC in the production function to $a = 0.4$.[22] The other parameter values are chosen such that the collateral and the quantity constraints start to bind in a range where there is still a non-negligible density of the investment shock. Furthermore, with these parameters, $\psi > 0$ and $\bar{k} > \Phi$, which means that we move from region 1 to region 3 and then to region 4 as $\varepsilon$ increases (see section 4.1 for a detailed discussion).

21 This is in the range of haircuts the Euro system applies to credit claims with a rating of BBB, see Table 4 in Bindseil et al. (2017).

22 This choice determines the location of the curves and thus the readability of the Figures whereas it does not affect the dynamics.

## 6.1 Simulations for different realisations of the investment shock

Figure 3 shows the demand for capital and the bank lending rate as a function of the investment shock for two different policy rates. To make the differences clearly visible we compare central bank lending rates of 2% (solid line) and 20% (dashed line). In region 1 where neither the collateral nor the quantity constraint bind, $k_C$ and $k_D$ increase linearly with increasing $\varepsilon$. Due to the parameter value chosen for the input share $a$, the demand for bank loans is higher than for CBDC loans. From the threshold $\varepsilon^{13}$ on the pledgeable collateral rises with a fraction $\eta$ of output, which is less than the efficient amount of investment would require. Consequently, both $k_C$ and $k_D$ continue to increase with the size of the project but with a lower slope than before. At $\varepsilon^{34}$ the quantity constraint $\bar{k}$ starts to bind, preventing a further increase in $k_C$. By contrast, $k_D$ increases with an even steeper slope since all additional investment has to be financed by bank loans.

A higher policy rate $\iota_C$ mainly dampens demand for $k_C$ with limited impact on $k_D$. With the lower demand for $k_C$, both the collateral and the quantity constraint start to bind only at higher values of $\varepsilon$.[23] The slight downward shift in $k_D$ in Figure 3 can be explained by the distortion caused by a higher $\iota_C$ that, as both types of capital are imperfect substitutes, also dampens the demand for $k_D$.

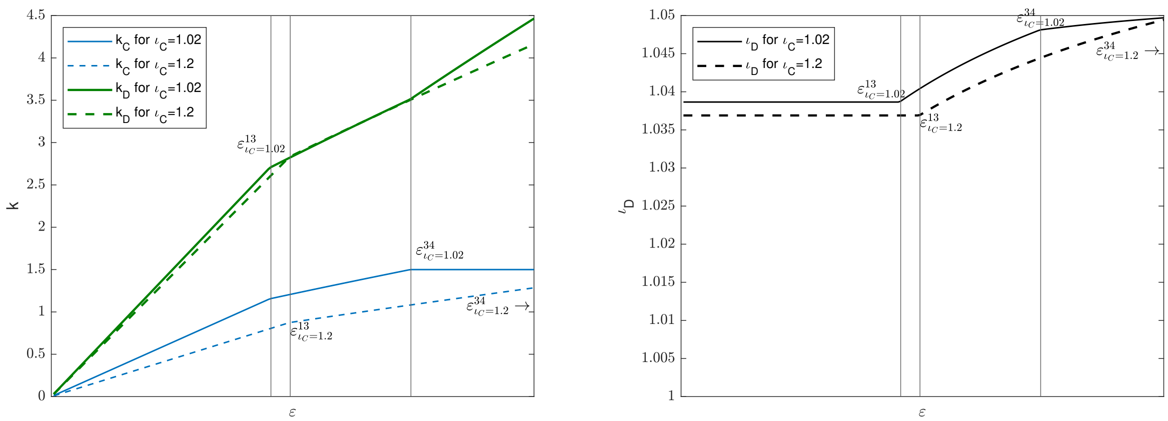



**Figure 3**
**Simulated $k_C, k_D$ and $\iota_D$ for different $\iota_C$**

The right panel in Figure 3 shows the corresponding bank lending rates, which are

23 With the parameter values chosen for our simulations, the quantity constraint does not bind over the range of $\varepsilon$ that we show in Figure 3.

determined by the banker's share in the total surplus over $k_D$, see (24). The key parameter determining the level of the bank lending rate is the banker's share in the total surplus, $\theta$, which we set to $\theta = 0.05$. A decline in the total surplus also lowers $\iota_D$, i.e. the share that the banker obtains as a remuneration for the $k_D$ loan. Note that $\iota_D$ exceeds the interest rate on CBDC for $\iota_C = 1.02$. With a binding collateral constraint, entrepreneurs invest less than the optimal amount of capital for the size of the investment shock they have drawn. Therefore, the marginal gain from an additional unit of invested capital increases with $\varepsilon$, leading to an increase in $\iota_D$, as the total surplus rises more quickly than $k_D$. In the unconstrained region the lending rate is independent of the size of the investment project. Once the quantity constraint starts to bind as well, all additional loans are made in inside money, $k_D$. Relative to the increase in the total surplus, more inside money is now invested, which leads to a decrease in the slope of $\iota_D$ in this region. The dashed line in Figure 3 shows that with a higher policy rate the lending rate on inside money decreases. At the first sight, this may seem counter-intuitive but the reason is that the higher cost entrepreneurs incur for their CBDC loans reduces the total surplus and thus translates into a lower profit for the bankers for the same size of an investment project.

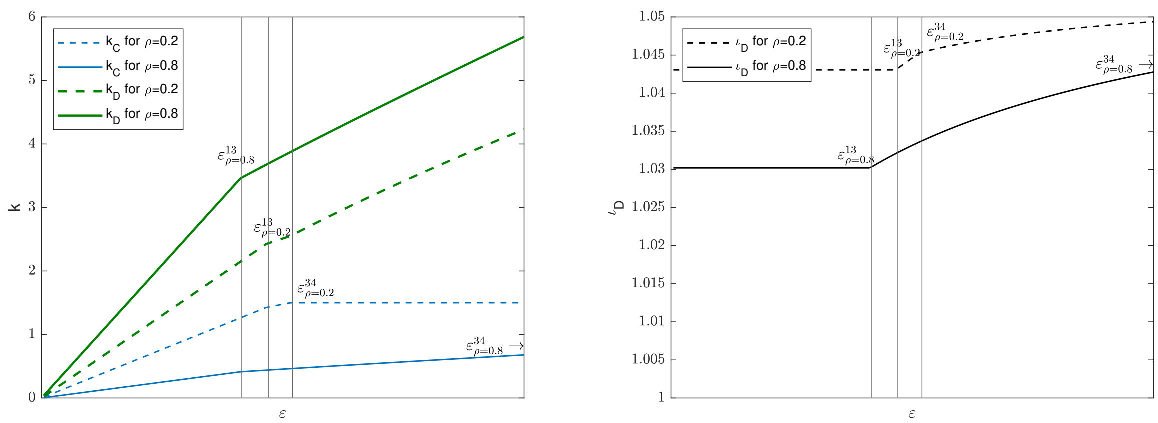



**Figure 4**
**Simulated $k_C, k_D$, and $\iota_D$ of $\varepsilon$ for different degrees of substitutability**

We next explore how $k_C$, $k_D$ and $\iota_D$ depend on the degree of substitution. While Figure 3 shows simulations for $\rho = 0.5$, Figure 4 plots $k_C$, $k_D$ and $\iota_D$ as a function of the investment shock $\varepsilon$ for $\rho = 0.2$ and $\rho = 0.8$. A low $\rho$ can be interpreted as CBDC having more cash-like features that make it less suitable as a substitute for bank deposits whereas a high $\rho$ could be seen as CBDC offering similar functionalities as bank deposits. Consequently, a high $\rho$ boosts demand for $k_D$ at the expense of $k_C$ (solid lines). Qualitatively, the evolution of $k_C$ and $k_D$ over the whole range if $\varepsilon$ shocks is similar independently of the values of $\rho$, except that when $\rho$ is relatively small, as in this case the demand for $k_C$ is higher.

The right panel in Figure 4 shows the impact of the substitution parameter, $\rho$, on the bank lending rate. As before, the lending rate starts to rise when the collateral constraint is binding. With a high degree of substitutability (solid line), the demand for $k_D$ is higher relative to $k_C$ which reduces the ratio of the total surplus to $k_D$ and, hence, also $\iota_D$.

## 6.2 Simulation of aggregated variables

To investigate the effects of the three policy tools – namely the policy rates $\iota_C$, the collateral haircut, $\eta$, and the quantity constraint $\bar{k}$ – on the capital allocation, bank lending rates and welfare, we aggregate all variables over the entire distribution of $\varepsilon$.

Invested capital, aggregated over the entire distribution, is defined as:

$$k_C^{aggr} = \int_0^\infty k_C(\varepsilon)dG(\varepsilon) \text{ and } k_D^{aggr} = \int_0^\infty k_D(\varepsilon)dG(\varepsilon) \ .$$

Accordingly, the average spread between deposit and lending rates for inside money is defined as:

$$\iota_D^{mean} = \int_0^\infty \iota_D(\varepsilon)dG(\varepsilon) = \int_0^\infty 1 + \frac{S_B(\varepsilon)}{k_D(\varepsilon)}dG(\varepsilon) = 1 + \theta \int_0^\infty \frac{TS(\varepsilon)}{k_D(\varepsilon)}dG(\varepsilon) \ .$$

Finally, welfare is calculated as the difference between output that can be consumed in the next period and invested capital, integrated over all investment shocks:

$$Welfare = \int_0^\infty \beta y(\varepsilon) - k(\varepsilon)dG(\varepsilon) = \int_0^\infty \varepsilon^{1-\alpha} f(k_C(\varepsilon), k_D(\varepsilon)) - k_C(\varepsilon) - k_D(\varepsilon)dG(\varepsilon) \ .$$

### 6.2.1 Different policy rates

Of particular interest is whether the demand for CBDC and the impact on bank intermediation can be steered by varying the interest rate on CBDC. The left panel of Figure 5 shows aggregate capital investment in inside money and CBDC as a function of the policy rate spread, $\iota_C$. As the latter increases and CBDC loans become more expensive, the demand for $k_C$ declines. The entrepreneurs' capital demand for inside money increases with $\iota_C$ if $\rho$ is relatively high, as the lower CBDC investment can more easily be substituted by inside money investment. In this case, the remuneration of CBDC is an effective instrument to affect bank intermediation which we interpret as the ratio between $k_D$ and $k_C$. If $\rho$ is small, however, both $k_D$ and $k_C$ decrease with increasing $\iota_C$, which means that the degree of disintermediation does not react strongly to changes in the remuneration of CBDC.[24] The right panel of Figure 5 shows the corresponding bank lending rates. For both values of $\rho$ they slightly decrease in $\iota_C$, since the bank lending rates reflect the banker's share of the total surplus, which becomes smaller when CBDC loans are more expensive. When $\rho$ is smaller (dashed line), less $k_D$ and more $k_C$ are used than with a higher $\rho$, corresponding to a higher $\iota_D$.

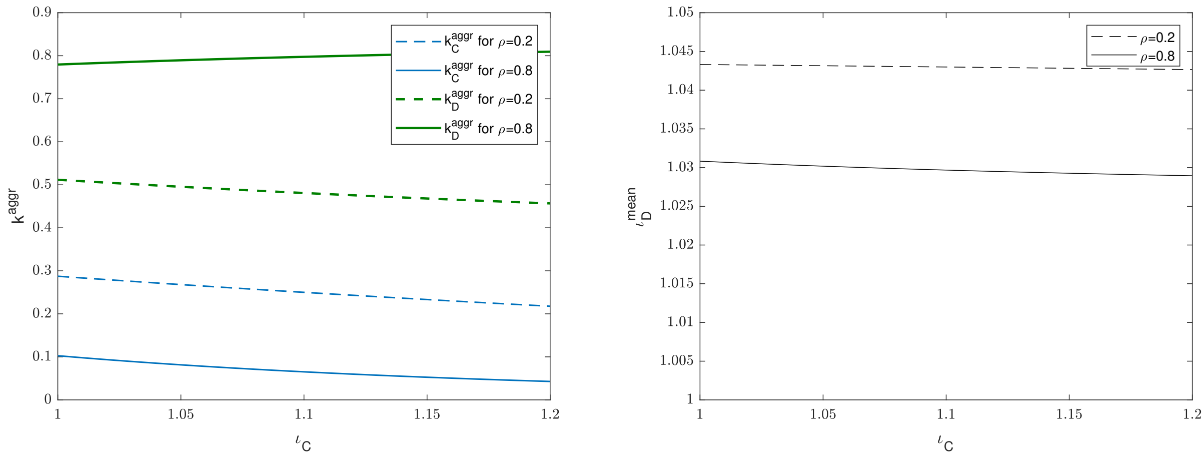



**Figure 5**
**Simulated aggregated capital, $k_C^{aggr}$ and $k_D^{aggr}$, and average bank lending rate, $\iota_D$, as a function of $\iota_C$ for different degrees of substitutability**

Figure 6 shows the evolution of welfare as a function of the policy rate, $\iota_C$. To maximize welfare, a social planner would choose $\iota_C = 1$, as this value eliminates the distortion introduced by the central bank's lending spread. An increase in the interest rate spread between the central bank's deposit and lending rate reduces welfare, the more so if entrepreneurs cannot easily switch from CBDC to inside money, i.e. when $\rho$ is small. With a higher degree of substitutability, welfare is less affected by high policy rates as entrepreneurs can more easily substitute away from $k_C$. At intermediate policy rates, however, welfare falls less quickly

24 Note that disintermediation occurs in relative terms, meaning that there is no abrupt disintermediation of $k_D$-loans at any specific interest rat $i_C$, as opposed to Keister and Sanches (2019).

with low substitutability. This is due to production taking place at a combination of $k_C$ and $k_D$ the ratio of which is closer to the optimal value and thus more efficient.

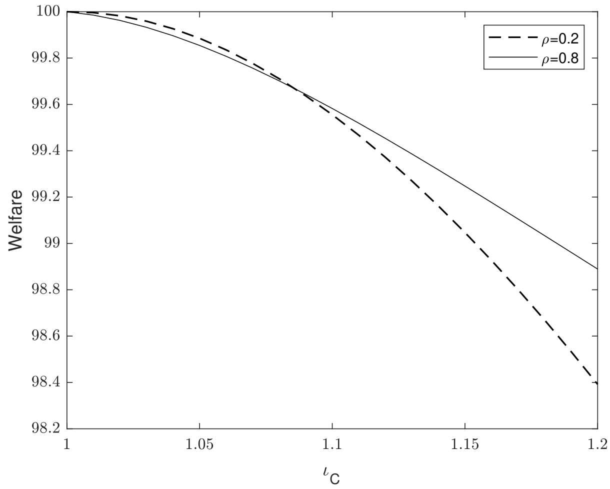



**Figure 6**
**Simulated welfare of $\iota_C$ for different degrees of substitutability**

### 6.2.2 Different collateral constraints

Besides the interest rate spread on CBDC, another tool that affects affects capital investment, bank lending rates and welfare is the collateral haircut. As illustrated in the left panel of Figure 7, capital increases with a higher share of pledgeable output, $\eta$. If $\rho$ is low and, hence, the two forms of capital are not easily substitutable, both $k_C$ and $k_D$ increase less with a higher $\eta$ because of the quantity constraint on CBDC. When $\rho$ is relatively high, however, $k_D$ continues to increase also when $\eta$ becomes high, since CBDC can be substituted by inside money relatively easily.

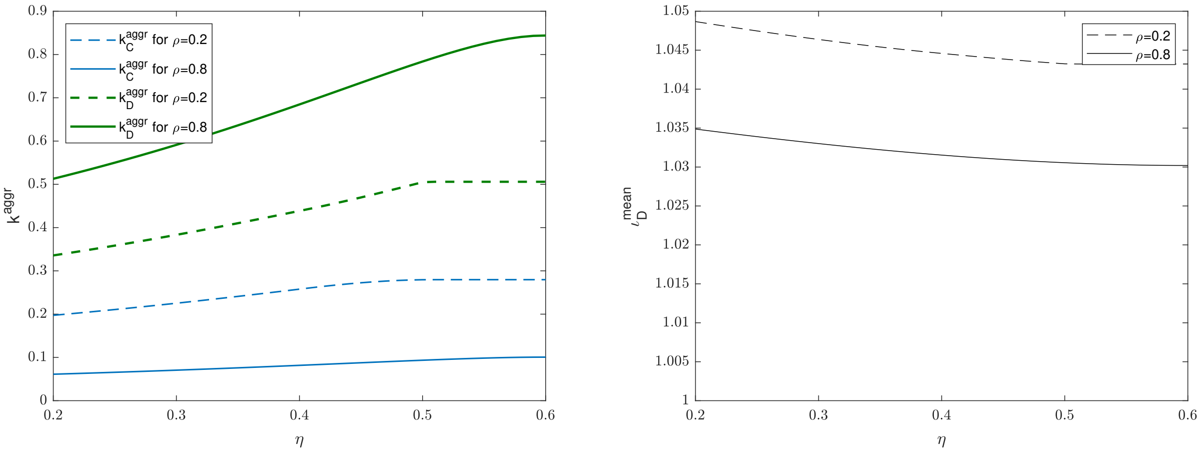



**Figure 7**
**Simulated aggregated capital, $k_C^{aggr}$ and $k_D^{aggr}$, and average bank lending rate $\iota_D$ as a function of $\eta$ for different degrees of substitutability**

The average bank lending rate decreases as $\eta$ increases, as can be seen in the right panel of Figure 7, since the total surplus increases less than invested inside money. With a higher $\eta$, fewer entrepreneurs are constrained in their capital investments and, therefore, more entrepreneurs are closer to their optimal capital allocation (as long as the quantity constraint does not bind), leading to a lower marginal surplus of an additional unit of capital. As inside money is less substitutable by CBDC when $\rho$ is smaller, less $k_D$ is invested for a given total surplus, and therefore, the bank lending rate is higher than with a high degree of substitutability.

Figure 8 shows that welfare increases with a higher share of pledgeable output, $\eta$. As $\eta$ increases, the amount of invested capital increases so that at some point the quantity constraint on CBDC starts to bind. Therefore, as $\eta$ increases, welfare increases with a smaller slope, more so when $\rho$ is rather small and CBDC can be substituted by inside money less easily. For any level of $\eta$, welfare is higher when the degree of substitution between inside money and CBDC is higher, since then entrepreneurs are more flexible in their capital allocation and can choose a combination that is closer to the optimum.

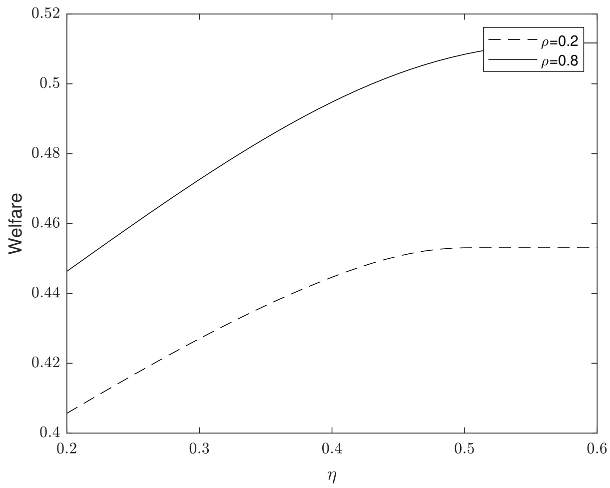



**Figure 8**
**Simulated welfare of $\eta$ for different degrees of substitutability**

### 6.2.3 Different quantity constraints

Finally, we study different quantity constraints on CBDC, $\bar{k}$ as another policy tool of the central bank to steer the amount of invested CBDC. Figure 9 (left panel) shows capital investment as a function of a cap on CBDC. As the latter increases the invested amount of CBDC increases. Inside money investments depend on the degree of substitutability, though. At low rates of substitution, the demand for inside money can even increase more quickly than the demand for CBDC as the quantity constraint is loosened. If the degree of substitutability is high, however, demand for inside money falls, reflecting disintermediation of banks given that policy rates are kept constant. From a certain level of $\bar{k}$ onward, bank intermediation seems to reach a steady state, which is reached faster, when $\rho$ is high and more inside money is used.

The right panel of Figure 9 shows that for a low $\rho$ the increase of $k_D$ corresponds to a decrease in the bank lending rate as the quantity constraint becomes more slack. When $\rho$ is high, these patterns are reversed. As the capital investments reach a constant level when $\bar{k}$ is high enough, the bank lending rates do so as well.

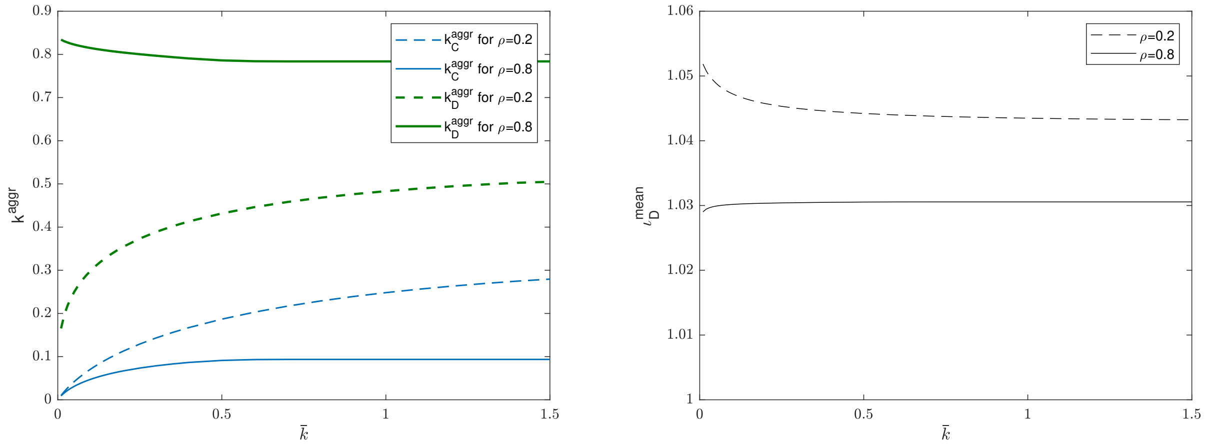



**Figure 9**
**Simulated aggregated capital, $k_C^{aggr}$ and $k_D^{aggr}$, and average bank lending rate $\iota_D$ as a function of $\bar{k}$ for different degrees of substitutability**

Figure 10 shows that raising the upper bound on CBDC leads to an increase in welfare, which is not surprising as the model by construction assigns an important role to CBDC. While the quantity constraint matters less for welfare if substitutability between $k_D$ and $k_C$ is high, welfare gains are more significant if inside and outside money are not easily substitutable, i.e. if $\rho$ is low.

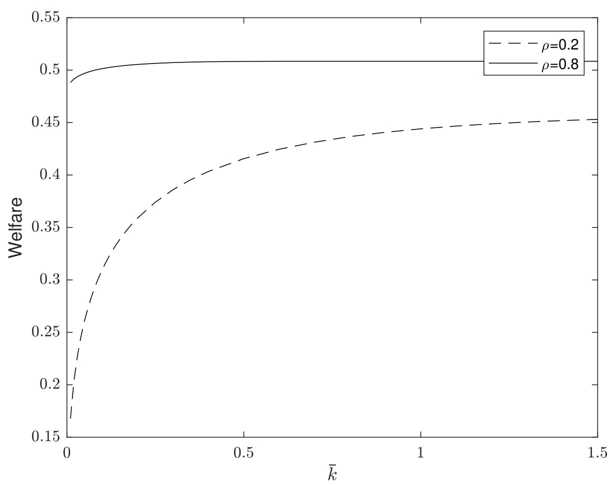



**Figure 10**
**Simulated welfare of $\bar{k}$ for different degrees of substitutability**

## 6.3 Welfare gains of CBDC and the degree of substitutability

Finally, we investigate how an increased prevalence of CBDC affects welfare. As it has become clear above, the answer depends crucially on the assumption for the degree of substitutability between CBDC and inside money. Therefore, we compare for a range of $\rho \in (0, 1)$ welfare obtained in the case of a high quantity constraint ($\bar{k}^{high}$) with welfare obtained in the case of a low quantity constraint ($\bar{k}^{low}$). The welfare gain of expanding CBDC supply as a function of $\rho$ is then calculated according to the following equation:[25]

$$\text{Welfare gain}(\rho) = \text{Welfare}(\bar{k}^{high}, \rho) - \text{Welfare}\ (\bar{k}^{low}, \rho)\ .$$

Figure 11 shows that the welfare gain of expanding CBDC supply is high for a low degree of substitutability and approaches 0 as $\rho$ becomes close to 1. When CBDC and inside money are not close substitutes (low $\rho$), welfare is much lower in a case where almost no CBDC can be used for investment than it is in a case where the quantity constraint only binds for large investment projects. Therefore, for low values of $\rho$, expanding CBDC supply leads to considerable welfare gains. In contrast, when these two forms of money can be substituted rather easily (high $\rho$), the increased supply of CBDC has almost no effect on welfare. The elasticity of substitution between CBDC and bank deposits will be affected by the design features of CBDC, such as its ease of use in specific situations. Our results suggest that the welfare gain from CBDC is small if CBDC covers similar use cases as bank deposits. If CBDC provides benefits to users in situations where bank deposits are less useful or not universally accepted, welfare gains from expanding CBDC supply are larger.

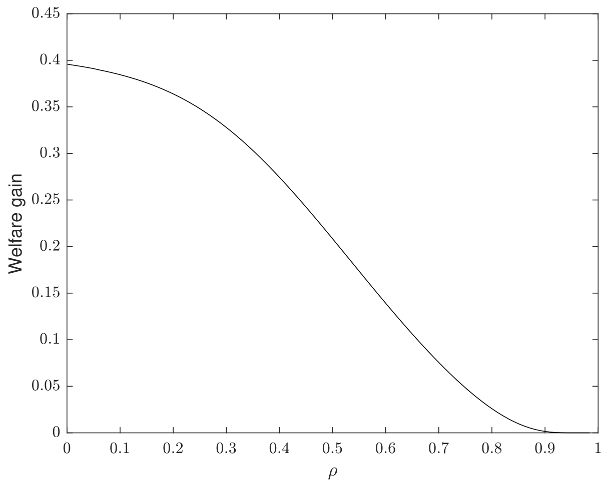



**Figure 11**
**Simulated welfare gain of introducing CBDC as a function of $\rho$**

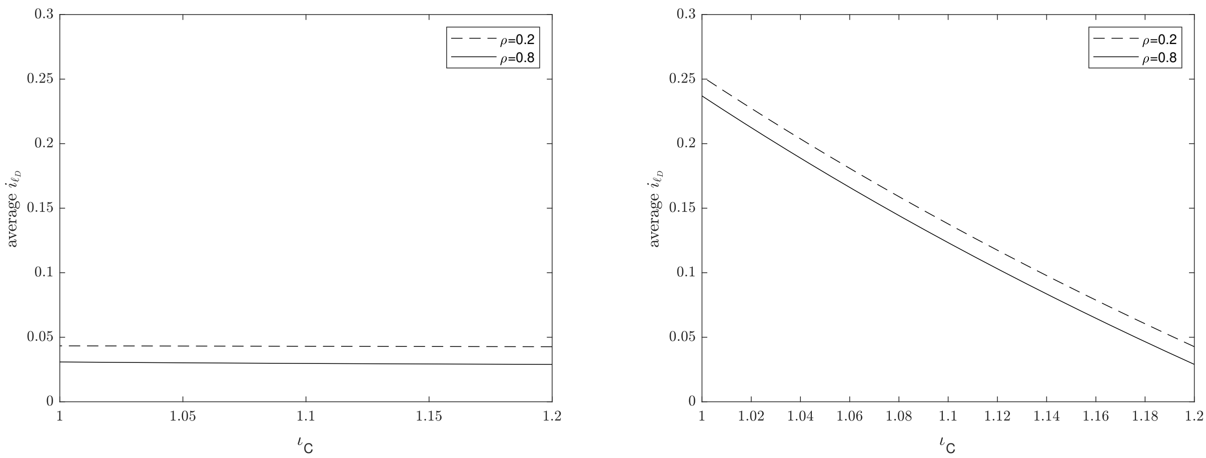
25 We set $\bar{k}^{high} = 1.5$ as in our baseline simulations and $\bar{k}^{low} = 0.001$.

## 6.4 Discussion of the assumptions on $i_D$

In this section, we explore how the assumptions on the bank deposit rate, $i_D$, affect the results for the bank loan rate, $i_{\ell_D}$. Until now, we have assumed that $i_D = 0$. We will compare this case with the case $i_D = i_C$. For both assumptions, we will analyse how a change in the policy rates, expressed in $\iota_C$, affects the inside money loan rate.

Figure 12 shows the bank loan rate, $i_{\ell_D}$, as a function of $\iota_C$ for assumption 1 (left panel) and assumption 2 (right panel). Recall that $\iota_C = \frac{1+i_{\ell_C}}{1+i_C}$ is the spread between the CBDC loan rate, $i_{\ell_C}$, and the CBDC deposit rate, $i_C$. For assumption 1, it does not matter whether a change in $\iota_C$ stems from a change in $i_C$ or $i_{\ell_C}$, since $i_D = 0$ by assumption. With $i_D = 0$, $i_{\ell_D} = \iota_D - 1$, and we can interpret $\iota_D$ as the (gross) bank loan rate. Therefore, the left panel in Figure 12 is basically the same as the right panel in Figure 5.

For assumption 2, $i_D = i_C$ (shown in the right panel of Figure 12), we need to distinguish, whether an increase in $\iota_C$ arises from an increase in $i_{\ell_C}$ or a reduction in $i_C$, since $i_D$ follows the latter. Let us assume that $i_{\ell_C} = 0.2$ is constant. That means that a gradual increase of $\iota_C$ from 1 to 1.2, as depicted in Figure 12, is equivalent to a gradual decrease in $i_C$ and, hence, generates a gradual decrease of $i_D$ from 0.2 to 0. Since the spread $\iota_D$ only decreases very little as $\iota_C$ changes (as it can be seen in Figure 5), $i_{\ell_D}$ decreases as $\iota_C$ increases. Differently speaking, as the central bank decreases the CBDC deposit rate and the bank deposit rate follows, the bank loan rate decreases as well, and even slightly more.




**(a)** $i_D = 1$

**(b)** $i_D = i_C$

**Figure 12**
**Average bank lending rate $i_{\ell_D}$ of $\iota_C$ for different degrees of substitutability**

Note that when $i_C = 0$ and, hence, $\iota_C = 1.2$, both graphs in Figure 12 display the same values for the bank lending rate, $i_{\ell_D}$, since in this case, $i_D = 0$ under both assumptions. We have only shown the bank lending rate in this section, since, as explained above, the real quantities and welfare are the same under the two assumptions. For these, only the value of $\iota_C$ – that is, only the spread between the CBDC loan rate and the CBDC deposit rate – matters.

# 7 Conclusion

We build a general equilibrium search model in which the central bank and commercial banks compete in supplying money to finance investment projects paid for in inside money or CBDC. Production and trade occur on different island economies, a set-up motivated by preferences for specific forms of payment. As entrepreneurs cannot commit to repay debt, workers require payment in the form of a generally accepted medium of exchange, a friction that gives rise to the demand for money. They borrow inside and outside money from commercial banks and from the central bank. But, because of the limited commitment friction, they are required to pledge collateral.

We define CBDC as a public or central bank liability that is interest bearing or paying and may be quantitatively constrained by the central bank. The provision of both inside and outside money is subject to a collateral constraint. The central bank fixes interest rates on CBDC and possible quantity or collateral constraints exogenously, banks and firms negotiate on bank lending rates and volumes.

We show that capital allocation and the welfare gains depend on the degree by which collateral or quantity constraints are binding, as well as on the spread between the CBDC deposit and lending rate. All three parameters constitute distortions lowering output. Conversely, relaxing CBDC caps and collateral constraints is strictly welfare improving. CBDC improves the overall allocation of resources and thereby increases output and welfare (as long as production inputs are not perfect substitutes), as it always reduces frictions in credit provision. At the same time, the provision of CBDC can reduce commercial bank credit and thereby disintermediate banks to some extent. Increasing the interest rate on

CBDC is effective in containing bank disintermediation, in particular if inside and outside money are close substitutes.

We leave the impact of CBDC on monetary policy transmission over the business cycle or on financial stability to future research. In our setting prices are flexible and money is neutral over the business cycle. In addition assets are safe and liquid. Our modelling framework is therefore informative of the impact of CBDC on the steady state, i.e. structural changes in the financial system.

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# Appendix: Proofs

***Derivation of the first-best quantities.*** In what follows we calculate the first-best investment quantities that would be attained in the absence of the limited commitment friction. Note that the disutility of producing capital goods occurs in period $t$ while the utility of consuming the generic good output occurs in period $t+1$. Hence the optimization problem satisfies

$$(k_C^*, k_D^*) = \arg\max_{k_C, k_D} \beta y - k_C - k_D = \varepsilon^{1-\alpha} f - k_C - k_D.$$

The first-order conditions for $k_D$ and $k_C$ satisfy

$$\varepsilon^{1-\alpha} f_C = 1 \text{ and } \varepsilon^{1-\alpha} f_D = 1. \tag{28}$$

Note that the ratio $k_C/k_D$ satisfies[26]

$$\omega^* \equiv \frac{k_C^*}{k_D^*} = \left(\frac{a}{1-a}\right)^{\frac{1}{1-\rho}}. \tag{29}$$

Then, from (29) we obtain $k_C^* = k_D^* \omega^*$. Use this expression to derive $f$ as a function of $k_D^*$ :

$$f = (k_D^*)^{\alpha}\,\Omega^*, \text{ where } \Omega^* \equiv \left[a\,(\omega^*)^{\rho} + 1 - a\right]^{\alpha/\rho}$$

Then, use $\varepsilon^{1-\alpha} f_D = 1$ to solve for $k_D^*$ :

$$k_D^* = \varepsilon^{\frac{1-\alpha}{1-\rho}} \alpha^{\frac{1}{1-\rho}} f^{\frac{\alpha-\rho}{\alpha(1-\rho)}} (1-a)^{\frac{1}{1-\rho}}$$

Substitute $f$ to get a closed-form solution for $k_D^*$:

$$k_D^* = \varepsilon \alpha^{\frac{1}{1-\alpha}} (\Omega^*)^{\frac{\alpha-\rho}{\alpha(1-\alpha)}} (1-a)^{\frac{1}{1-\alpha}}$$

Use $k_D^*$ to get a closed-form solution for $f$:

$$f = \varepsilon^{\alpha} \alpha^{\frac{\alpha}{1-\alpha}} (\Omega^*)^{\frac{1-\rho}{1-\alpha}} (1-a)^{\frac{\alpha}{1-\alpha}}$$

The total surplus $TS = \beta y^* - k_C^* - k_D^*$ satisfies:

$$TS^* = \beta y^* (1-\alpha)$$

Thus, $\beta y^* - k_C^* - k_D^* \geqslant 0$ for all $\varepsilon$. To see this, consider (29) and write it as follows:

$$(1-a)\,\omega^* = a\,(\omega^*)^{\rho}$$

26 To see this, note that $\frac{f_C}{f_D} = \frac{\alpha f^{\frac{\alpha-\rho}{\alpha}} a k_C^{\rho-1}}{\alpha f^{\frac{\alpha-\rho}{\alpha}} (1-a) k_D^{\rho-1}} = 1$.

We can use this expression to obtain:

$$\begin{aligned}\Omega^* &= [(1-a)\,\omega^* + 1 - a]^{\alpha/\rho} \\ &= (1-a)^{\alpha/\rho}\,[1+\omega^*]^{\alpha/\rho},\end{aligned}$$

implying that

$$\begin{aligned}k_D^* + k_C^* &= k_D^*\,(1+\omega^*) \\ &= k_D^*(\Omega^*)^{\rho/\alpha}(1-a)^{-1} \\ &= \varepsilon\alpha^{\frac{1}{1-\alpha}}\,(1-a)^{\frac{\alpha}{1-\alpha}}\,(\Omega^*)^{\frac{1-\rho}{1-\alpha}} \\ &= \alpha\beta y.\end{aligned}$$

In summary, the optimal quantities and the total surplus satisfy:

$$\begin{aligned}\beta y^* &= \varepsilon\alpha^{\frac{\alpha}{1-\alpha}}\,(1-a)^{\frac{\alpha}{1-\alpha}}\,(\Omega^*)^{\frac{1-\rho}{1-\alpha}} \\ k_D^* &= \varepsilon\alpha^{\frac{1}{1-\alpha}}\,(1-a)^{\frac{1}{1-\alpha}}\,(\Omega^*)^{\frac{\alpha-\rho}{\alpha(1-\alpha)}} \\ k_C^* &= k_D\omega^*. \\ TS^* &= \beta y^*\,(1-\alpha)\end{aligned}$$

■

***Derivation of the collateral constraint.*** The collateral constraint satisfies

$$(1+i_{\ell_C})\,\ell_C + (1+i_{\ell_D})\,\ell_D \leqslant e + \eta\varepsilon^{1-\alpha} f\,(k_C, k_D)\,. \tag{30}$$

If the entrepreneur borrows $\ell_C$ at the standing facility, then he has to pay back $(1+i_{\ell_C})\,\ell_C$. In real terms that quantity satisfies $(1+i_{\ell_C})\,\ell_C\phi_C^+$. Finally, to convert it into utility we have to discount it and obtain

$$\beta\ell_C\phi_C^+\,(1+i_{\ell_C})\,.$$

Using the workers' first-order condition, we can write this quantity as $\iota_C k_C$.

The banker's surplus is $S_B = \theta TS = \beta\phi_D^+\ell_D(1+i_{\ell_D}) - \beta\phi_D^+\ell_D(1+i_D)$. To guarantee this surplus, the banker requires that the entrepreneur pays back $(1+i_{\ell_D})\,\ell_D$. In real terms that quantity satisfies $(1+i_{\ell_D})\,\ell_D\phi_D^+$. Finally, to convert it into utility we have to discount it and obtain

$$\beta\ell_C\phi_D^+\,(1+i_{\ell_D})\,.$$

Using the workers' first-order condition, we can write this quantity as $\iota_D k_D$.

Consider, next, the right-hand side of the collateral constraint. The entrepreneur is endowed with an idiosyncratic endowment which yields utility $e/\beta$ when consumed in the settlement market. Since the collateral constraint is applied in the investment market, we need to discount the utility and so the discounted consumption utility of the idiosyncratic endowment is $e$.

The entrepreneur can also pledge the fraction $\eta$ of the output $y = \beta^{-1}\varepsilon^{1-\alpha} f(k_C, k_D)$. The quantity $y$ is also the consumption utility generated from consuming $y$. We need again to discount the consumption utility. Hence, the discounted utility of the pledgable output is

$$\eta\beta y = \eta\varepsilon^{1-\alpha} f(k_C, k_D).$$

Optimality requires that (14) holds with equality. That is, $(1-\theta)TS = S_E$, or equivalently $S_B = \theta TS$. Furthermore, from (12), we have

$$S_B = k_D \frac{(i_{\ell_D} - i_D)}{1+i_D} = k_D \frac{(1+i_{\ell_D} - (1+i_D))}{1+i_D} = \iota_D k_D - k_D.$$

Since $S_B = \theta TS$, we have

$$\iota_D k_D = \theta TS + k_D.$$

Use this expression to rewrite the collateral constraint (30) as follows:

$$\iota_C k_C + k_D + \theta TS \leqslant e + \eta\varepsilon^{1-\alpha} f(k_C, k_D).$$

■

***Proof of Lemma 1.*** In what follows, we impose parameter restrictions such that for $\varepsilon^{1-\alpha} f_C = \iota_C$ and $\varepsilon^{1-\alpha} f_D = 1$, we have

$$\iota_C k_C + k_D + \theta TS > \eta\varepsilon^{1-\alpha} f(k_C, k_D). \tag{31}$$

This restriction on parameters implies that for $e = 0$, all entrepreneurs' collateral constraints are binding. It then follows that for $e > 0$ entrepreneurs with a small $\varepsilon$ are unconstrained and entrepreneurs with large $\varepsilon$ are constrained.

To derive the critical value $\varepsilon$[13] we need to derive $k_C$ and $k_D$ under the assumption that constraints are not binding. Then, from (25) we get $k_C = k_D\omega$. As before, we can use this expression to derive $f(k_C, k_D)$ as a function of $k_D$:

$$f(k_C, k_D) = k_D^{\alpha}\Omega, \text{ where } \Omega \equiv [a\omega^{\rho} + 1 - a]^{\alpha/\rho} \text{ with } \omega = \frac{a}{\iota_C(1-a)}^{\frac{1}{1-\rho}}.$$

Following the same steps as before, we obtain

$$k_D = \varepsilon\alpha^{\frac{1}{1-\alpha}} \Omega^{\frac{\alpha-\rho}{\alpha(1-\alpha)}} (1-a)^{\frac{1}{1-\alpha}}.$$

Using $k_D$, we obtain an explicit solution for $f$ and $\beta y$ :

$$\begin{aligned} f &= k_D^{\alpha}\Omega = \varepsilon^{\alpha}\alpha^{\frac{\alpha}{1-\alpha}}\Omega^{\frac{1-\rho}{1-\alpha}}\left(1-a\right)^{\frac{\alpha}{1-\alpha}} \\ \beta y &= \varepsilon^{1-\alpha}f = \varepsilon\alpha^{\frac{\alpha}{1-\alpha}}\Omega^{\frac{1-\rho}{1-\alpha}}\left(1-a\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Also for the other variables, we can follow the same approach as in the derivations for the first-best solution. The only difference is that now $\omega = \left(\frac{a}{\iota_C(1-a)}\right)^{\frac{1}{1-\rho}}$ with $\iota_C$ possibly larger than 1. In summary, the equilibrium quantities in region 1), in which none of the constraint is binding, yet, satisfy

$$\beta y = \varepsilon\alpha^{\frac{\alpha}{1-\alpha}}\Omega^{\frac{1-\rho}{1-\alpha}}\left(1-a\right)^{\frac{\alpha}{1-\alpha}}, \tag{32}$$

$$k_D = \varepsilon\alpha^{\frac{1}{1-\alpha}}\Omega^{\frac{\alpha-\rho}{\alpha(1-\alpha)}}\left(1-a\right)^{\frac{1}{1-\alpha}}, \tag{33}$$

$$k_C = k_D\omega = \varepsilon\alpha^{\frac{1}{1-\alpha}}\Omega^{\frac{\alpha-\rho}{\alpha(1-\alpha)}}\left(1-a\right)^{\frac{1}{1-\alpha}}\omega, \tag{34}$$

$$TS = \left(1-\alpha\right)\beta y, \tag{35}$$

$$k_D + \iota_C k_C = \alpha\beta y. \tag{36}$$

We now use these quantities to derive an explicit condition such that $\iota_C k_C + k_D + \theta TS > \eta\varepsilon^{1-\alpha}f\left(k_C, k_D\right)$. Use (32)-(36) to get

$$\alpha + \theta(1-\alpha) > \eta.$$

Hence, the condition is

$$\psi \equiv \left(1-\theta\right)\alpha - \left(\eta-\theta\right) > 0.$$

Note that $\alpha > \eta$ is a sufficient condition for $\psi > 0$.

Finally, the critical value $\varepsilon^{13}$ solves

$$\iota_C k_C + k_D + \theta TS = e + \eta\beta y.$$

where $k_C$, $k_D, \beta y$, and $TS$ solve (32)-(35). Thus,

$$\alpha\beta y + \theta\left(1-\alpha\right)\beta y = e + \eta\beta y.$$

Solving for $\beta y$ yields

$$\beta y = \frac{e}{\left(1-\theta\right)\alpha - \left(\eta-\theta\right)}.$$

Replacing $\beta y$ yields

$$\varepsilon^{13} = \frac{e}{\left[\left(1-\theta\right)\alpha - \left(\eta-\theta\right)\right]\left[\alpha^{\frac{\alpha}{1-\alpha}}\Omega^{\frac{1-\rho}{1-\alpha}}\left(1-a\right)^{\frac{\alpha}{1-\alpha}}\right]}. \tag{37}$$

Note that $\varepsilon^{13}$ is increasing in $\iota_C$ since $\Omega$ is decreasing in $\iota_C$. Note further that output at $\varepsilon^{13}$ satisfies

$$y^{13} = \frac{e}{(1-\theta)\,\alpha - (\eta - \theta)},$$

which is independent of $\iota_C$. This implies that in the region where entrepreneurs are unconstrained, output decreases as $\iota_C$ increases. ■

***Proof of Proposition 1***. We first derive the critical values, $\varepsilon^{13}, \varepsilon^{34}, \varepsilon^{12},$ and $\varepsilon^{24}$, and then the condition under which the collateral constraint binds earlier than the quantity constraint.

**Collateral constraints binds first.** In this case, there are two critical values: $\varepsilon^{13}$ (moving from region 1 to region 3) and $\varepsilon^{34}$ (moving from region 3 to 4).

The critical value $\varepsilon^{13}$ is given in (37). The critical value $\varepsilon^{34}$ and the corresponding quantities $k_C^{34}$ and $k_D^{34}$ can be derived by solving the following equations simultaneously:

$$\begin{aligned}\frac{k_C^{34}}{k_D^{34}} &= \left(\frac{a}{\iota_C\,(1-a)}\right)^{\frac{1}{1-\rho}} \text{ with } k_C^{34} = \bar{k} \text{ and}\\ \iota_C k_C^{34} + k_D^{34} + \theta TS &= e + \eta \left(\varepsilon^{34}\right)^{1-\alpha} f.\end{aligned}$$

**Quantity constraints binds first.** Assume now that, as we increase $\varepsilon$ from $\varepsilon = 0$, we move from 1) to 2) and then to 4). There are two critical values: $\varepsilon^{12}$ (moving from region 1 to region 2) and $\varepsilon^{24}$ (moving from region 2 to 4).

From (34), the critical value $\varepsilon^{12}$ solves $\bar{k} = \varepsilon^{12}\alpha^{\frac{1}{1-\alpha}}\Omega^{\frac{\alpha-\rho}{\alpha(1-\alpha)}}\,(1-a)^{\frac{1}{1-\alpha}}\,\omega$, leading to

$$\varepsilon^{12} = \frac{\bar{k}}{\alpha^{\frac{1}{1-\alpha}}\Omega^{\frac{\alpha-\rho}{\alpha(1-\alpha)}}\,(1-a)^{\frac{1}{1-\alpha}}\,\omega}.$$

The critical value $\varepsilon^{24}$ and the corresponding quantities $k_C^{24}$ and $k_D^{24}$ solve

$$\begin{aligned}\varepsilon^{1-\alpha} f_D &= 1 \text{ with } k_C^{24} = \bar{k} \text{ and}\\ \iota_C k_C^{34} + k_D^{34} + \theta TS &= e + \eta \left(\varepsilon^{34}\right)^{1-\alpha} f.\end{aligned}$$

**Which constraint binds first?** Whether the collateral constraint or the quantity constraint binds first, depends on the critical values $\varepsilon^{12}$ and $\varepsilon^{13}$. If $\varepsilon^{12} > \varepsilon^{13}$, the collateral constraint binds first. We have

$$\begin{aligned}\varepsilon^{13} &= \frac{e}{\left[(1-\theta)\,\alpha - (\eta-\theta)\right]\alpha^{\frac{\alpha}{1-\alpha}}\Omega^{\frac{1-\rho}{1-\alpha}}\,(1-a)^{\frac{\alpha}{1-\alpha}}} \text{ and}\\ \varepsilon^{12} &= \frac{\bar{k}}{\alpha^{\frac{1}{1-\alpha}}\Omega^{\frac{\alpha-\rho}{\alpha(1-\alpha)}}\,(1-a)^{\frac{1}{1-\alpha}}\,\omega}.\end{aligned}$$

Hence, $\varepsilon^{12} > \varepsilon^{13}$ if

$$\begin{aligned}
\frac{\bar{k}}{\alpha^{\frac{1}{1-\alpha}} \Omega^{\frac{\alpha-\rho}{\alpha(1-\alpha)}} (1-a)^{\frac{1}{1-\alpha}} \omega} &> \frac{e}{\left[(1-\theta)\alpha - (\eta-\theta)\right] \alpha^{\frac{\alpha}{1-\alpha}} \Omega^{\frac{1-\rho}{1-\alpha}} (1-a)^{\frac{\alpha}{1-\alpha}}} \\
\bar{k} &> \frac{e\omega\alpha(1-a)}{\left[(1-\theta)\alpha - (\eta-\theta)\right] \Omega^{\frac{\rho}{\alpha}}} \\
\bar{k} &> \frac{e(1-a)\omega\alpha}{\left[(1-\theta)\alpha - (\eta-\theta)\right] \left[a\omega^{\rho} + 1 - a\right]} \\
\bar{k} &> \frac{e(1-a)\omega\alpha}{\left[(1-\theta)\alpha - (\eta-\theta)\right] \left[\omega\iota_C(1-a) + 1 - a\right]} \\
\bar{k} &> \frac{e(1-a)\omega\alpha}{\psi(1-a)(\omega\iota_C + 1)} \\
\bar{k} &> \frac{e\alpha}{\psi(\iota_C + \frac{1}{\omega})} \equiv \Phi \text{ .}
\end{aligned}$$

■

***Effects of CBDC interest rates***. Here we derive the derivatives $\frac{dk_D}{d\iota_C}$ and $\frac{dk_C}{d\iota_C}$.

If unconstrained, we have

$$\begin{aligned}
k_D &= \varepsilon\alpha^{\frac{1}{1-\alpha}} (1-a)^{\frac{1}{1-\alpha}} \left[a\omega^{\rho} + 1 - a\right]^{\frac{\alpha-\rho}{\rho(1-\alpha)}} . \\
&= \varepsilon\alpha^{\frac{1}{1-\alpha}} (1-a)^{\frac{1}{1-\alpha}} \left[(1-a)\iota_C\omega + 1 - a\right]^{\frac{\alpha-\rho}{\rho(1-\alpha)}} \\
&= \varepsilon\alpha^{\frac{1}{1-\alpha}} (1-a)^{\frac{\alpha}{\rho(1-\alpha)}} \left[\iota_C\omega + 1\right]^{\frac{\alpha-\rho}{\rho(1-\alpha)}} ,
\end{aligned}$$

since $\omega = \left(\frac{a}{\iota_C(1-a)}\right)^{\frac{1}{1-\rho}}$, implying that $\omega\iota_C(1-a) = a\omega^{\rho}$. Next, note that $\iota_C\omega = \left(\frac{a}{(\iota_C)^{\rho}(1-a)}\right)^{\frac{1}{1-\rho}}$ and, hence, $\frac{d\iota_C\omega}{d\iota_C} < 0$.

The closed form solution can be derived as follows:

$$\begin{aligned}
\frac{dk_D}{d\iota_C} &= \varepsilon\alpha^{\frac{1}{1-\alpha}} (1-a)^{\frac{1}{1-\alpha}} \frac{\alpha-\rho}{\rho(1-\alpha)} \left[a\omega^{\rho} + 1 - a\right]^{\frac{\alpha-\rho}{\rho(1-\alpha)}-1} a\rho\omega^{\rho-1} \\
&\quad \frac{1}{1-\rho} \left(\frac{a}{\iota_C(1-a)}\right)^{\frac{1}{1-\rho}-1} \frac{-(1-a)a}{(\iota_C(1-a))^2} \\
\frac{dk_D}{d\iota_C} &= \frac{\frac{\alpha-\rho}{\rho(1-\alpha)} k_D}{\left[a\omega^{\rho} + 1 - a\right]} a\rho\omega^{\rho-1} \frac{1}{1-\rho} \frac{\omega\iota_C(1-a)}{a} \frac{-(1-a)a}{(\iota_C(1-a))^2} \\
\frac{dk_D}{d\iota_C} &= -\frac{(\alpha-\rho) k_D a\omega^{\rho}}{(1-\alpha)\left[a\omega^{\rho} + 1 - a\right](1-\rho)\iota_C} \lesseqgtr 0
\end{aligned}$$

Note that $\frac{d\iota_C\omega}{d\iota_C} < 0$ implies that $\frac{dk_D}{d\iota_C} > 0$ if $\rho > \alpha$.

The effect on $k_C$ is always negative:

$$\begin{aligned}
\frac{dk_C}{d\iota_C} &= \frac{dk_D}{d\iota_C}\omega + k_D\frac{d\omega}{d\iota_C} = -\frac{(\alpha-\rho)\,k_D a\omega^\rho\omega}{(1-\alpha)\left[a\omega^\rho+1-a\right](1-\rho)\,\iota_C} - k_D\frac{1}{1-\rho}\left(\frac{a}{\iota_C\,(1-a)}\right)^{\frac{1}{1-\rho}}\frac{1}{\iota_C} \\
&= -\frac{(\alpha-\rho)\,k_D a\omega^\rho\omega}{(1-\alpha)\left[a\omega^\rho+1-a\right](1-\rho)\,\iota_C} - \frac{k_D\omega}{(1-\rho)\,\iota_C} \\
&= \frac{\omega k_D}{(1-\rho)\,\iota_C}\left[-\frac{(\alpha-\rho)\,a\omega^\rho}{(1-\alpha)\left[a\omega^\rho+1-a\right]} - 1\right] \\
&= -\frac{\omega k_D}{(1-\rho)\,\iota_C}\left[\frac{(\alpha-\rho)\,a\omega^\rho + (1-\alpha)\left[a\omega^\rho+1-a\right]}{(1-\alpha)\left[a\omega^\rho+1-a\right]}\right] \\
&= -\frac{\omega k_D}{(1-\rho)\,\iota_C}\left[\frac{a\omega^\rho(1-\rho) + (1-\alpha)\,(1-a)}{(1-\alpha)\left[a\omega^\rho+1-a\right]}\right] < 0
\end{aligned}$$

Let us now analyze the effect of $\iota_C$ on $\iota_D$. From the banker's surplus, we have:

$$\iota_D = \theta\frac{TS}{k_D} + 1 = \theta\frac{\varepsilon^{1-\alpha} f\,(k_C, k_D) - \iota_C k_C - k_D}{k_D} + 1 = \theta\left[\frac{\varepsilon^{1-\alpha} f\,(k_C, k_D) - \iota_C k_C}{k_D} - 1\right] + 1,$$

which, using the explicit forms derived in the appendix, can be written as:

$$\begin{aligned}
\iota_D = \quad & \theta\left[\varepsilon^{1-\alpha}k_D^{\alpha-1}\Omega - \iota_C\omega - 1\right] + 1 = \theta\left[\varepsilon^{1-\alpha}\left(\varepsilon\alpha^{\frac{1}{1-\alpha}}\Omega^{\frac{\alpha-\rho}{\alpha(1-\alpha)}}(1-a)^{\frac{1}{1-\alpha}}\right)^{\alpha-1}\Omega - \iota_C\omega - 1\right] + 1 \\
= \quad & \theta\left[\alpha^{-1}(1-a)^{-1}\Omega^{\frac{\rho}{\alpha}} - \iota_C\omega - 1\right] + 1 = \theta\left[\alpha^{-1}(1-a)^{-1}[a\omega^\rho + 1 - a] - \iota_C\omega - 1\right] + 1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d\iota_D}{d\iota_C} = \quad & \theta\frac{a}{\alpha(1-a)}\rho\omega^{\rho-1}\frac{1}{1-\rho}\left(\frac{a}{\iota_C(1-a)}\right)^{\frac{1}{1-\rho}-1}\frac{-a(1-a)}{(\iota_C(1-a))^2} \\
- \quad & \theta\left(\omega + \iota_C\frac{1}{1-\rho}\left(\frac{a}{\iota_C(1-a)}\right)^{\frac{1}{1-\rho}-1}\frac{-a(1-a)}{(\iota_C(1-a))^2}\right) \\
= \quad & \theta\left[\frac{a}{\alpha(1-a)}\omega^{\rho-1}\frac{\rho}{1-\rho}\omega\frac{-(1-a)}{\iota_C(1-a)} - \left(\omega + \iota_C\frac{1}{1-\rho}\omega\frac{-(1-a)}{\iota_C(1-a)}\right)\right] \\
= \quad & \theta\left[\frac{1}{\alpha}\frac{\rho}{\rho-1}\omega - \frac{\rho}{\rho-1}\omega\right] \\
= \quad & \theta\frac{1-\alpha}{\alpha}\frac{\rho}{\rho-1}\omega < 0 \text{ with a steeper slope for high } \rho \text{ and low } \iota_C.
\end{aligned}$$

■